# 8. Controllability

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# The Key Points of This Section

- we can compute minimum energy inputs so that $x(T) = x_{\text{des}}$

- we can measure controllability by looking at the SVD of the matrix

$$\begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix}$$

  for large $T$

- the singular values and left singular vectors give us the *controllability ellipsoid*, which tell us strong and weak directions in the state-space

- to compute this as $T \to \infty$, we solve the Lyapunov equation

$$W - AWA^T = BB^T$$

  the eigenvectors of $W$ are the axis directions and lengths of the controllability ellipsoid (over infinite time)

# Controlling the State

discrete-time LDS, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = 0$$

look at state at time $T$

$$x(T) = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T-1) \end{bmatrix}$$

ask *control* questions:

- find input sequence $u(0), \dots, u(T-1)$ so that $x(T) = x_{\text{des}}$
- find all input sequences that result in $x(T) = x_{\text{des}}$
- among all those, find the smallest, most efficient one

# how to control the state

$$x(T) = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T \begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}$$

minimum norm solution is $\begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T^{\dagger} x_{\text{des}}$

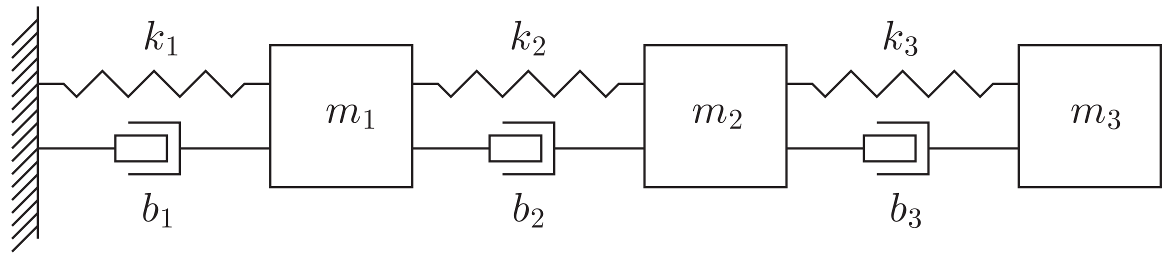
among all input sequences for which $x(T) = x_{\text{des}}$, this one has the smallest norm;

i.e, it minimizes

$$\sum_{t=0}^{T-1} \|u(t)\|^2 = \|u(0)\|^2 + \|u(1)\|^2 + \cdots + \|u(T-1)\|^2$$

called the *input energy*

# example: mass-spring system



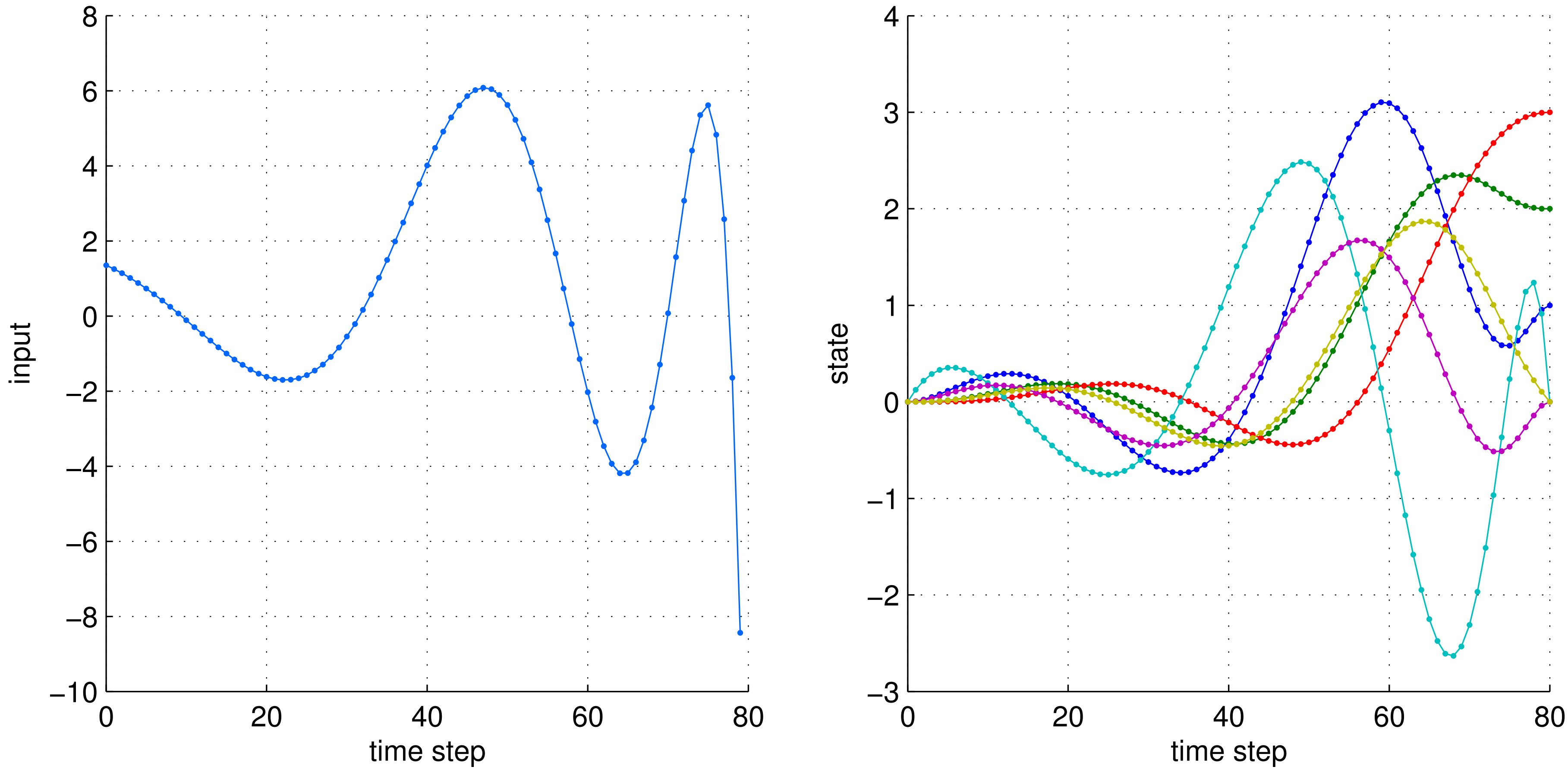

masses $m_i = 1$, spring constants $k = 1$, damping constants $b = 0.8$

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & 0.8 & 0 \\ 1 & -2 & 1 & 0.8 & -1.6 & 0.8 \\ 0 & 1 & -1 & 0 & 0.8 & -0.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$u(t)$ is force applied to mass 1

$x_{\text{des}} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$ at time step $T = 80$.

# example: control of spring-mass system




sampling period $h = 0.1$, optimal input achieves desired state

$$x(80) = x_{\mathsf{des}} = \begin{bmatrix}1 & 2 & 3 & 0 & 0 & 0\end{bmatrix}^T$$

## what next?

- what about assumptions? $H_T$ is fat and full rank when?
- computation size; we need to compute $H_T^\dagger$
- how does solution behave as we change $T$?

# Reachable Set

- we need $H_T$ to be fat

  $H_T$ has dimensions $n \times Tm$, so if $Tm < n$ we will have $\mathrm{range}(H_T) \neq \mathbb{R}^n$

- we need $H_t$ to be full rank

  again, if not, then $\mathrm{range}(H_T) \neq \mathbb{R}^n$

- $\mathrm{range}(H_T)$ is the set of states which are *reachable* at time $T$
- $\mathrm{range}(H_t) \subset \mathrm{range}(H_s)$ if $t \leq s$, so we can reach more points given more time

  because

$$H_T = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix}$$

# Cayley-Hamilton

characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A) \\ &= a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 \end{aligned}$$

note that $a_n \neq 0$

Cayley-Hamilton theorem

> $A$ satisfies its own characteristic equation

that is

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

important consequence: $A^n \in \operatorname{span}\{I, A, A^2, \ldots, A^{n-1}\}$

## controllability

we have

$$H_T = \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix}$$

for $t \geq n$, we can express $A^t$ as a linear combination of $A^{n-1}, A^{n-2}, \dots, A, I$

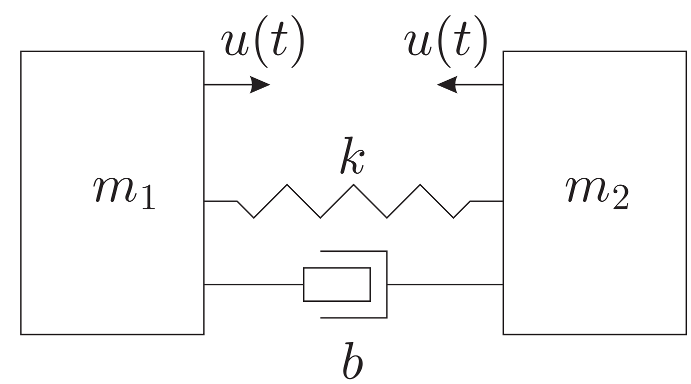
so if $T \geq n$, then

$$\boxed{\operatorname{range}(H_T) = \operatorname{range}(H_n)}$$

- if we are interested in reaching $x_{\text{des}}$ in time $T \geq n$, we just need to look at $H_n$ to determine the reachable set
- $H_n$ is called the *controllability matrix*
- if $\operatorname{range}(H_n) = \mathbb{R}^n$ the system is called *controllable* or *reachable*. sometimes we just say $(A, B)$ is reachable

## example: controlling identical masses

- identical masses connected by spring and damper
- input is tension force between masses
- masses, spring, damping constants $= 1$




continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u(t)$$

- starting from $x(0) = 0$, which states can we reach?

# example: controlling identical masses

discretization $\qquad A_d = e^{Ah} \qquad B_d = \int_0^h e^{As} B\, ds$

discrete-time controllability matrix $\qquad H_n = \begin{bmatrix} A_d^3 B_d & A_d^2 B_d & A_d B_d & B_d \end{bmatrix}$

compute reachable set $= \mathrm{range}(H_n)$ using SVD $\qquad H_n = U \Sigma V^T$

we find

$$\mathrm{range}(H_n) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

can reach those states with $x_1 = -x_2$ and $\dot{x}_1 = -\dot{x}_2$

because tension force does not change center of mass, or total momentum

# energy required

pseudo-inverse

$$H_T^\dagger = H_T^T (H_T H_T^T)^{-1}$$

minimum norm input sequence achieving $x(T) = x_{\text{des}}$ is

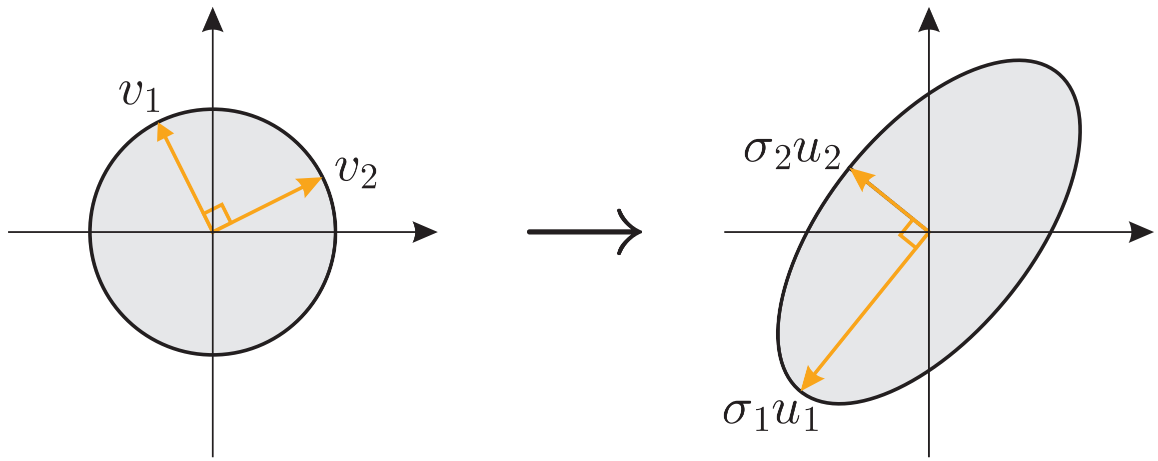
$$\begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix} = H_T^\dagger x_{\text{des}}$$

input energy required

$$\begin{aligned} \sum_{t=0}^{T} \|u(t)\|^2 &= x_{\text{des}}^T (H_T^\dagger)^T H_T^\dagger x_{\text{des}} \\ &= x_{\text{des}}^T (H_T H_T^T)^{-1} x_{\text{des}} \end{aligned}$$

# controllability ellipsoid

look at the set of states reachable at time $T$ with energy $$\sum_{t=0}^{T} \|u(t)\|^2 \leq 1$$




- *controllability ellipsoid:*
  semi-axis directions are left singular vectors of $H_T$
  semi-axis lengths are singular values of $H_T$
- short axis is *weakly controllable* direction
- *practical method* for determining controllability;
  gives a quantitative answer, not just yes or no

## controllability ellipsoid

singular values and left singular vectors are eigenvalues and eigenvectors of $H_T H_T^T$

call this matrix $W_T$

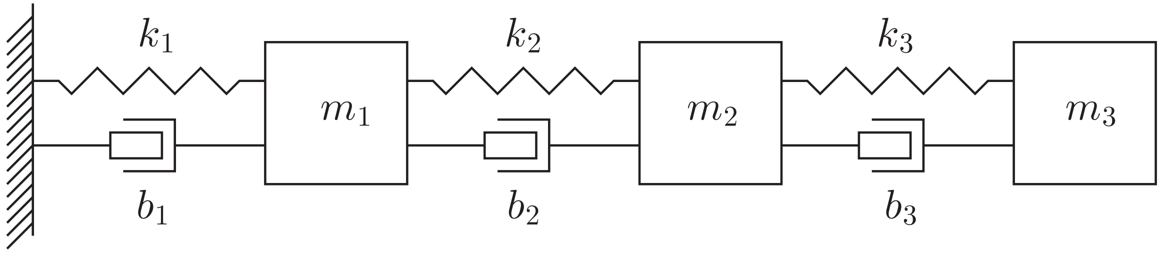
$$\begin{aligned} W_T &= \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} A^{T-1}B & A^{T-2}B & \dots & AB & B \end{bmatrix}^T \\ &= \sum_{k=0}^{T-1} A^k B B^T (A^T)^k \end{aligned}$$

- energy required to reach $x_{\text{des}}$ at time $T$ is $x_{\text{des}}^T W_T^{-1} x_{\text{des}}$
- if $t \geq s$ then $W_t \geq W_s$ so $W_t^{-1} \leq W_s^{-1}$, so

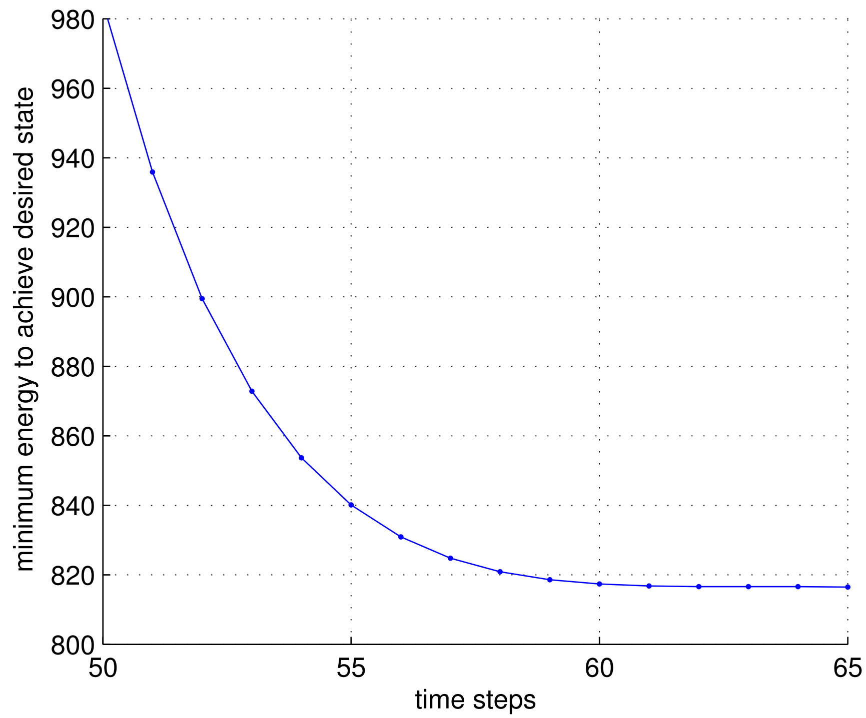
$$x_{\text{des}}^T W_t^{-1} x_{\text{des}} \leq x_{\text{des}}^T W_s^{-1} x_{\text{des}}$$

less energy is required to reach $x_{\text{des}}$ given more time

# example: energy against time




energy required



# infinite time problems

what happens as $T \to \infty$?

let

$$\begin{aligned} W &= \lim_{T\to\infty} W_T \\ &= \lim_{T\to\infty} \sum_{k=0}^{T} A^k B B^T (A^T)^k \end{aligned}$$

this converges if $\rho(A) < 1$, that is the system is stable.

- $W$ converges exponentially
- eigenvalues of $W$ measure amount of energy required to reach corresponding eigen-vectors with no restriction on how long it takes
- good practical measure of controllability
- how to compute it?

# Lyapunov Equation

if $A$ is stable, i.e., $\rho(A) < 1$, then

$$W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

we have

$$\begin{aligned} AWA^T &= \sum_{k=1}^{\infty} A^k B B^T (A^T)^k \\ &= W - BB^T \end{aligned}$$

$W$ satisfies the *Lyapunov equation*

$$\boxed{W - AWA^T = BB^T}$$

# example: solving Lyapunov equations

they are just linear equations, so it's easy

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

we want to solve $W - AWA^T = BB^T$, i.e.,

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

write this as

$$\begin{bmatrix} w_{11} \\ w_{21} \\ w_{12} \\ w_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{8} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{21} \\ w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

which is just a set of linear equations in $w_{ij}$

## solving Lyapunov equations

$$W = \begin{bmatrix} w_1 & \dots w_n \end{bmatrix} \qquad BB^T = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}$$

we can write $W - AWA^T = BB^T$ as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} - \begin{bmatrix} a_{11}A & a_{12}A & \dots & a_{1n}A \\ a_{21}A & & & \\ \vdots & & & \vdots \\ a_{n1}A & a_{n2}A & \dots & a_{nn}A \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

this is $n^2$ linear equations in $n^2$ unknowns

## solutions of Lyapunov equations

we know that if

$$W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

then $W$ satisfies the equation

$$W - AWA^T = BB^T$$

key question: is there always a solution to this equation?

is it unique?

## solutions of Lyapunov equations

let $Q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}$ and $\hat{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$

write the Lyapunov equation $W - AWA^T = Q$ as

$$(I - \hat{A})\hat{w} = \hat{q}$$

if $\rho(A) < 1$ then for any $Q \in \mathbb{R}^{n \times n}$ we have a solution to $W - AWA^T = Q$ which is

$$W = \sum_{k=0}^{\infty} A^k Q (A^T)^k$$

so $\operatorname{range}(I - \hat{A}) = \mathbb{R}^{n^2}$. i.e,

> if $\rho(A) < 1$ then $(I - \hat{A})$ is invertible

## solutions of Lyapunov equations

so we have

> if $\rho(A) < 1$ then the solution $W$ to $W - AWA^T = BB^T$ is unique

we can solve this to determine *controllability*; it tells us

- the ellipsoid

$$\left\{ x \in \mathbb{R}^n \mid x^T W^{-1} x \leq 1 \right\}$$

  is the set of states reachable with input energy $\displaystyle\sum_{t=0}^{\infty} \|u(t)\|^2 \leq 1$

- the corresponding eigenvectors of $W$ tell us the strongly and weakly controllable directions