9. Observability

- Estimating the initial state
- Observability
- Example: observability
- Least-squares observers
- The observability ellipsoid
- Infinite time
- Computing observability
- Example: one mass attached to two springs
- Estimating other states
- Example: estimating other states

The Key Points of This Section

- once we know $u(0), \ldots, u(T-1)$ and $y(0), \ldots, y(T-1)$, we just have a linear equation relating the data to the initial state $x(0)$

- so we can use least squares to estimate it
- and we can compute the *estimation ellipsoid*
- which tells how sensitive the estimate is to errors

- the infinite-time case is easy to compute via a *Lyapunov equation*
- which gives a *practical measure* of observability

Estimating the Initial State

discrete-time system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

which means

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{T-2}B & CA^{T-3}B & \ldots & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} + \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} x(0)$$

we'd like to estimate $x(0)$; ask *estimation* questions

- given $u(0), \ldots, u(T-1)$ and $y(0), \ldots, y(T-1)$, find $x(0)$
- find *all* $x(0)$ consistent with measured data
- if there is no exactly consistent $x(0)$, find an approximate one

estimating the initial state

this is just

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} = P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} + J_T x(0)$$

where

$$P_T = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{T-2}B & CA^{T-3}B & \dots & CB & D \end{bmatrix} \qquad J_T = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix}$$

- P_T is block Toeplitz; when $x(0) = 0$ it maps the input sequence to the output sequence
- J_T maps initial state to output sequence

estimating the initial state

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} = J_T x(0)$$

- this is the usual estimation problem of the form '$y_{\text{meas}} = Ax$', where

  y_{meas} is the LHS of the equation above

  x is $x(0)$

  A is J_T

- we need

  - more measurements than unknowns; need J_T to be skinny
  - $\text{null}(J_T) = \{0\}$ i.e. J_T must have full rank

  otherwise we will have *ambiguity* in solution for $x(0)$

observability

- if $z_1, z_2 \in \mathbb{R}^n$, and $z_1 - z_2 \in \mathrm{null}(J_T)$, then initial states $x(0) = z_1$ and $x(0) = z_2$ are *indistinguishable* given $u(0), \dots, u(T-1)$ and $y(0), \dots, y(T-1)$.

- if $T \geq n$, then by the *Cayley-Hamilton theorem*,

$$\mathrm{null}(J_T) = \mathrm{null}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} = \mathrm{null}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- which means: if we can determine $x(0)$ from $u(0), \dots, u(T-1), y(0), \dots, y(T-1)$, then we can determine it from $u(0), \dots, u(n-1), y(0), \dots, y(n-1)$

- $\mathrm{null}(J_n)$ is called the *unobservable subspace*

  J_n is called the *observability matrix*

  if $\mathrm{null}(J_n) = \{0\}$ we call the system *observable*

example: observability

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x(t)
\end{aligned}
$$

then

$$
J_n = J_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 5 \end{bmatrix}
$$

$\operatorname{rank}(J_2) = 2$ so the system is observable

Least-Squares Observers

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix} = J_T x(0)$$

so we can estimate $x(0)$ by

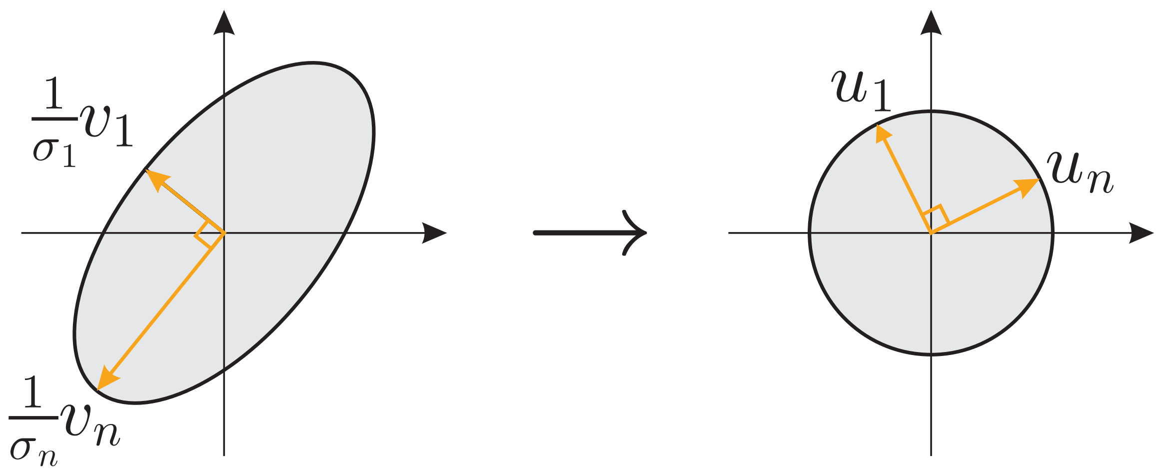
$$x_{\text{est}}(0) = J_T^{\dagger} \left(\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(T-1) \end{bmatrix}_{\text{measured}} - P_T \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(T-1) \end{bmatrix}_{\text{measured}} \right)$$

this gives the exact solution if one exists

Observability Ellipsoid

when $u(0) = u(1) = \cdots = u(T-1) = 0$, we have $\begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix} = J_T x(0)$

which states give *output energy* $\displaystyle\sum_{t=0}^{T-1} \|y(t)\|^2 \le 1$



- semi-axis directions given by right-singular vectors of J_T

  semi-axes lengths given by $\dfrac{1}{\sigma_i(J_T)}$

observability ellipsoid

when the input is zero,

$$\sum_{t=0}^{T-1} \|y(t)\|^2 = \begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix}^T \begin{bmatrix} y(0) \\ \vdots \\ y(T-1) \end{bmatrix}$$

$$= x(0)^T J_T^T J_T x(0)$$

observability ellipsoid is

$$\left\{ x \in \mathbb{R}^n \mid x^T J_T^T J_T x \leq 1 \right\}$$

- eigenvalues of $J_T^T J_T$ are squared singular values of J_T
  eigenvectors of $J_T^T J_T$ are right singular vectors of J_T

- long axes: *weakly observable* directions
  short axes: *strongly observable* directions

observability ellipsoid

let

$$\begin{aligned} V_T &= J_T^T J_T \\ &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}^T \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix} \\ &= \sum_{k=0}^{T-1} (A^T)^k C^T C A^k \end{aligned}$$

so if $t \geq s$ then $V_t \geq V_s$, i.e.,

- observability ellipsoid is smaller at larger t
- states are more observable given more data

observability ellipsoid over infinite time

to measure *practical observability*, look at infinite-time case

$$V = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

sum converges if $\rho(A) < 1$; then V satisfies

$$\boxed{V - A^T V A = C^T C}$$

called observability Lyapunov equation

computing observability

just like controllability:

- solution of Lyapunov equation $V - A^TVA = C^TC$ is easy; just linear equations
  solution is unique if $\rho(A) < 1$

- infinite time observability ellipsoid is

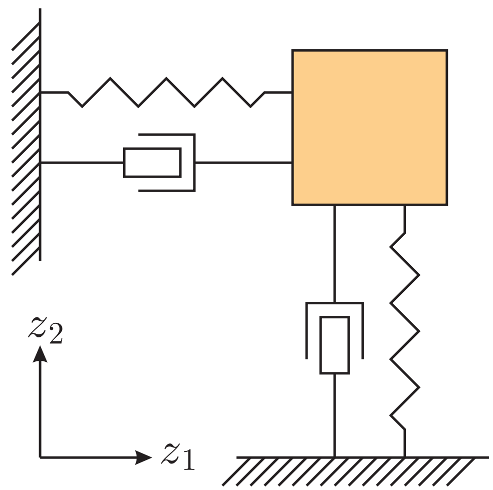
$$\left\{ x \in \mathbb{R}^n \mid x^TVx \leq 1 \right\}$$

  large eigenvalues means corresponding direction is strongly observable

- V is called *observability Gramian*

example: initial state observation

- unit mass on frictionless surface
- connected to two springs arranged perpendicularly
- applied force $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



linearized equations of motion are

$$\ddot{z}_1 + b_1 \dot{z}_1 + k_1 z_1 = u_1$$
$$\ddot{z}_2 + b_2 \dot{z}_2 + k_2 z_2 = u_2$$

example: state equations

- spring constants $k_1 = 0.1$, $k_2 = 0.5$
- damping constants $b_1 = 0.4$, $b_2 = 0.3$
- states $x(t) = \begin{bmatrix} z_1(t) & z_2(t) & \dot{z}_1(t) & \dot{z}_2(t) \end{bmatrix}^T$

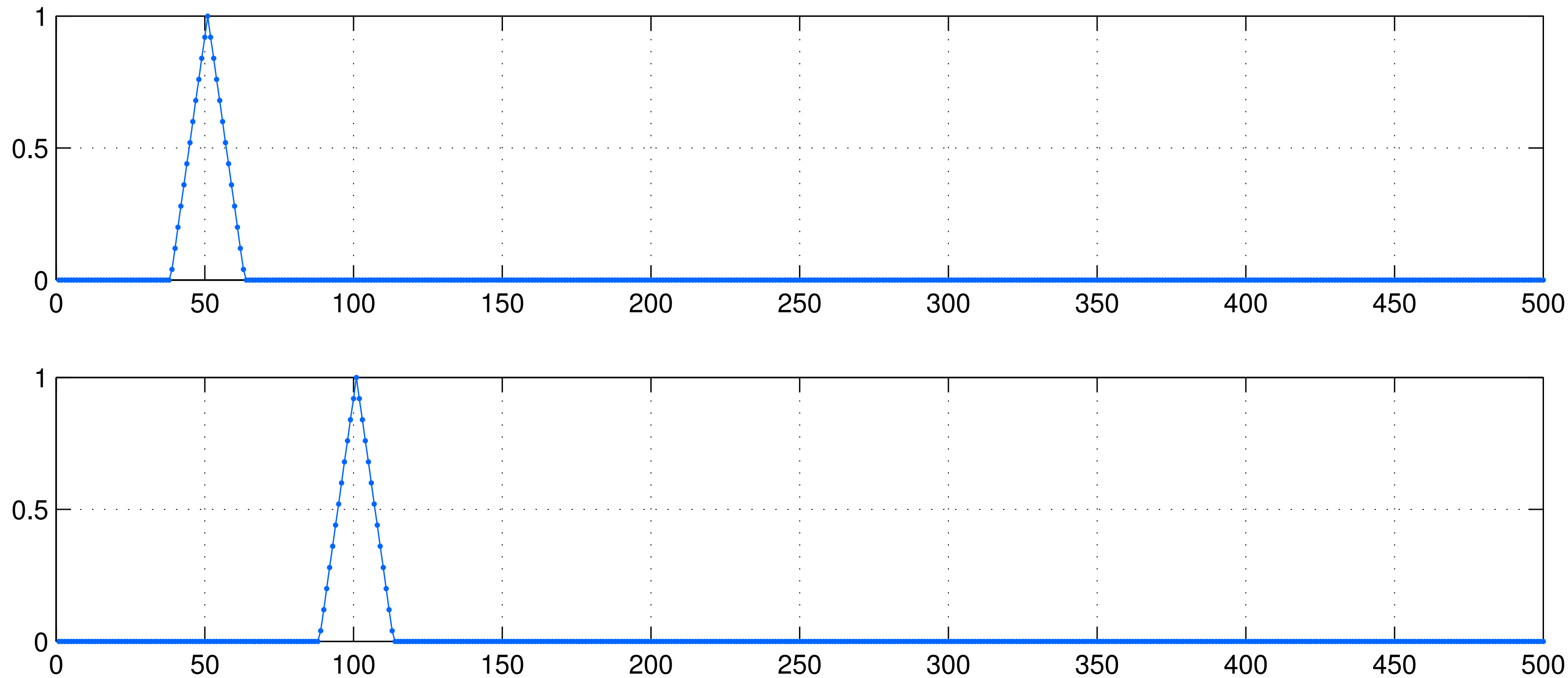
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & 0 & -b_1 & 0 \\ 0 & -k_2 & 0 & -b_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} x(t)$$

discretize, sampling period $h = 0.2$ gives

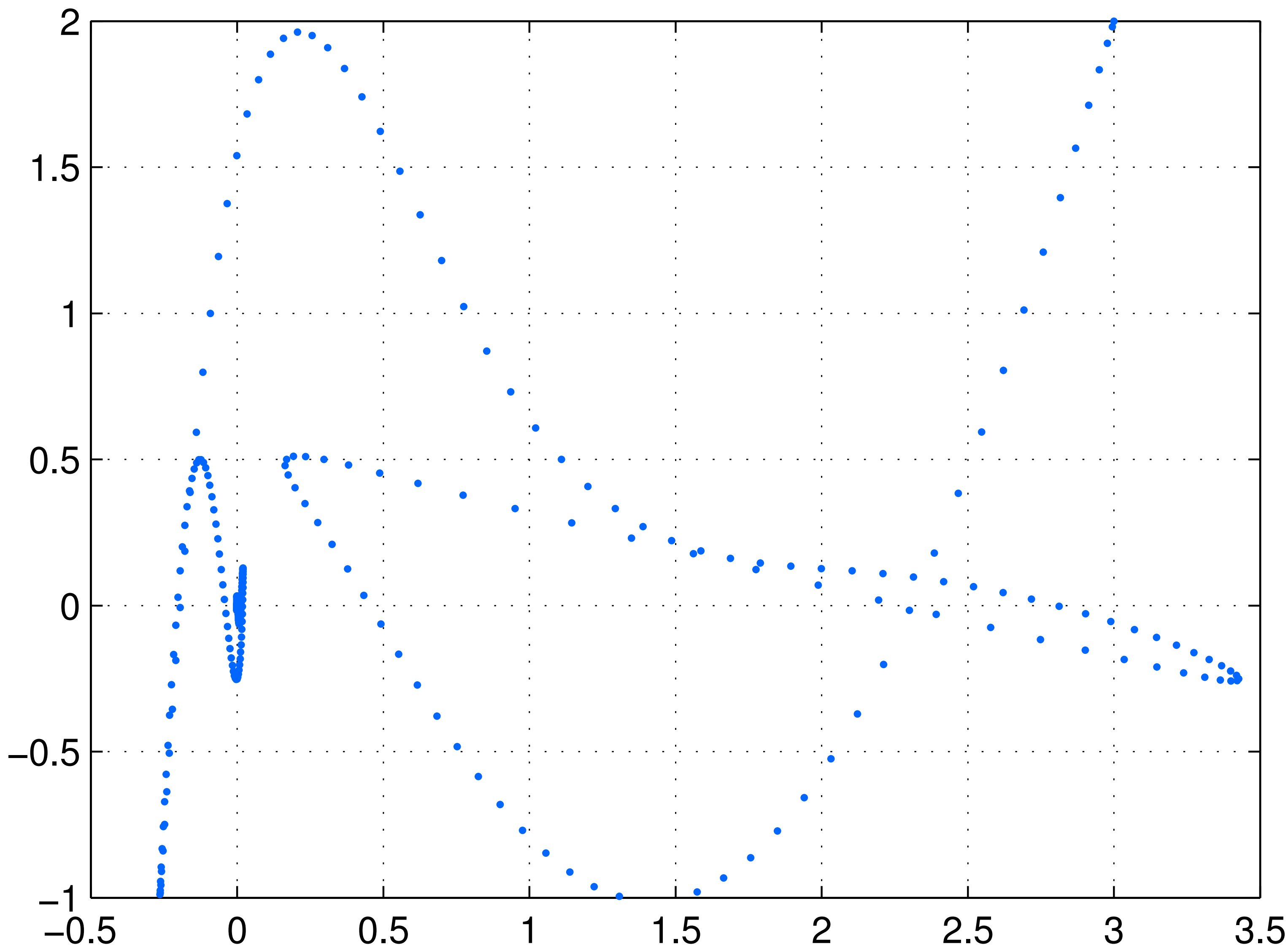
$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

example: input signals



example: trajectory

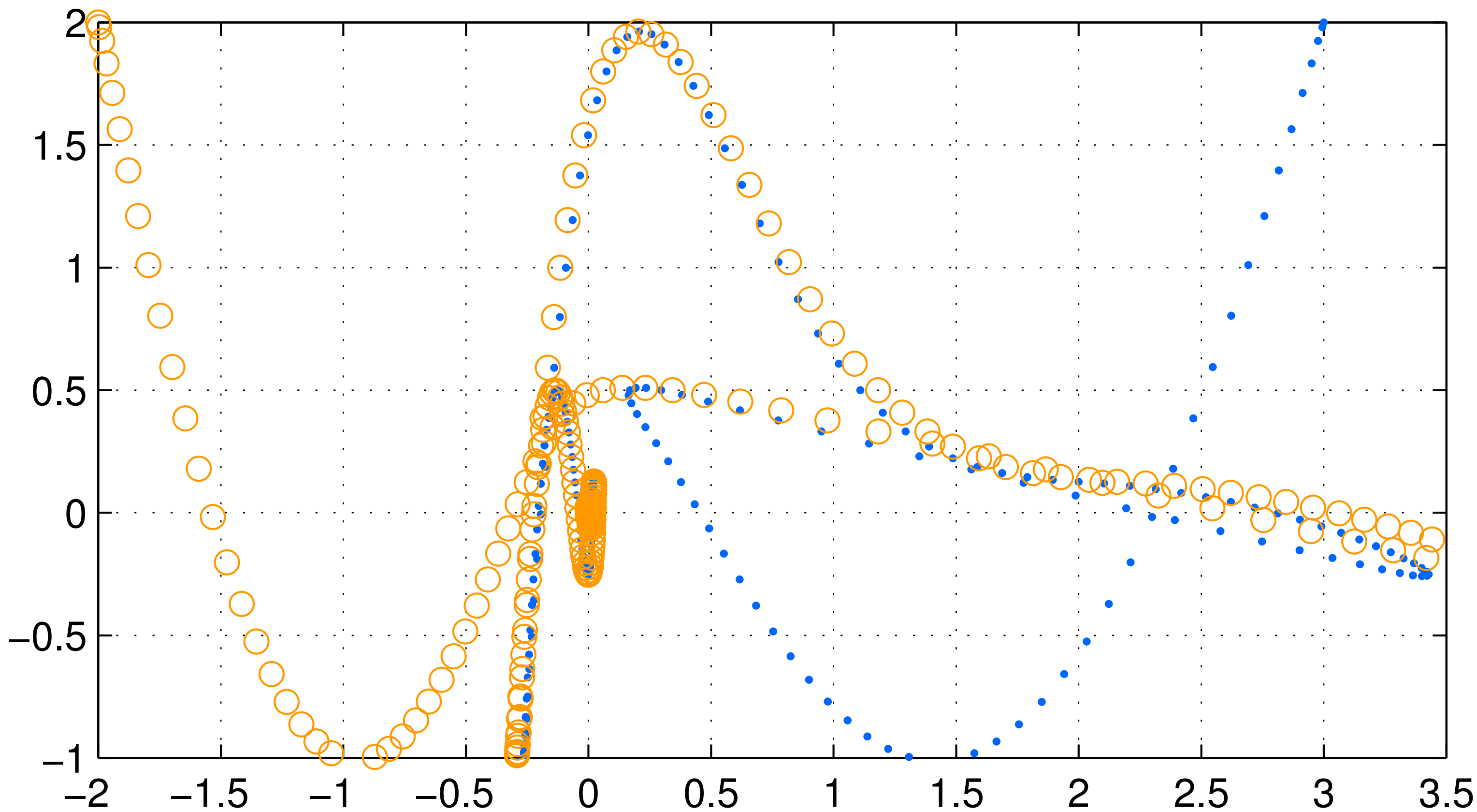
initial state $x(0) = \begin{bmatrix} 3 & 2 & 0 & 0 \end{bmatrix}^T$



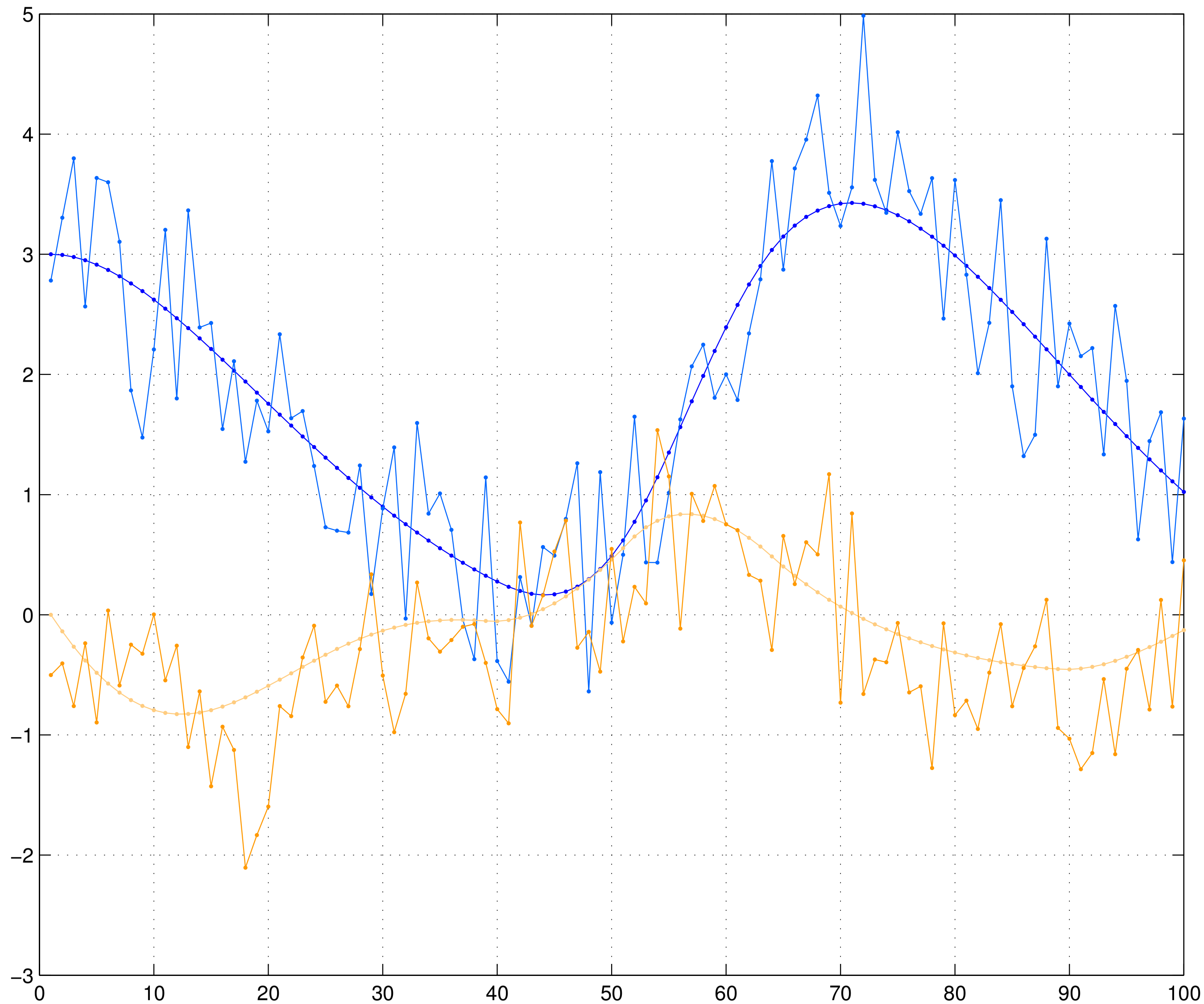
example: fading effect of initial conditions

initial state $x(0) = \begin{bmatrix}3 & 2 & 0 & 0\end{bmatrix}^T$

another initial state $\begin{bmatrix}-2 & 2 & 0 & 0\end{bmatrix}^T$



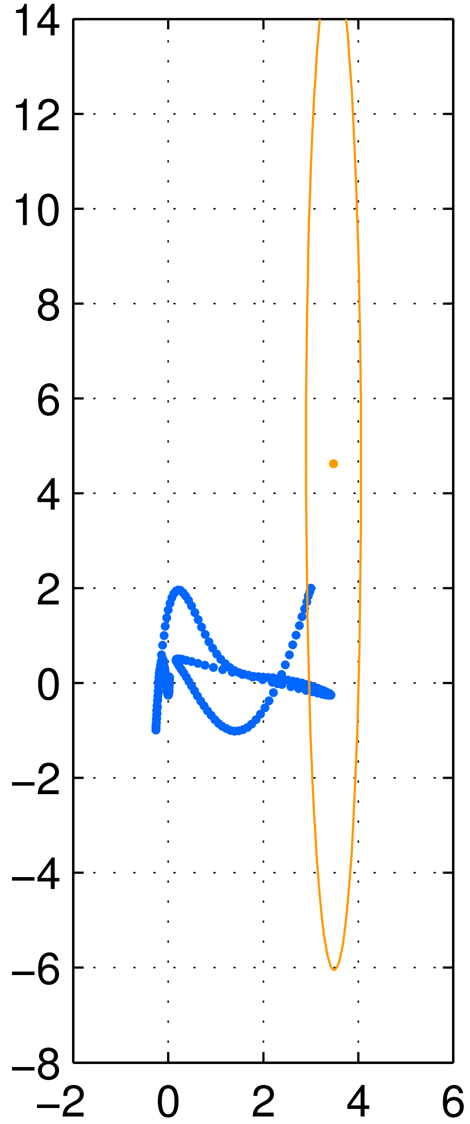
example: noisy measurements

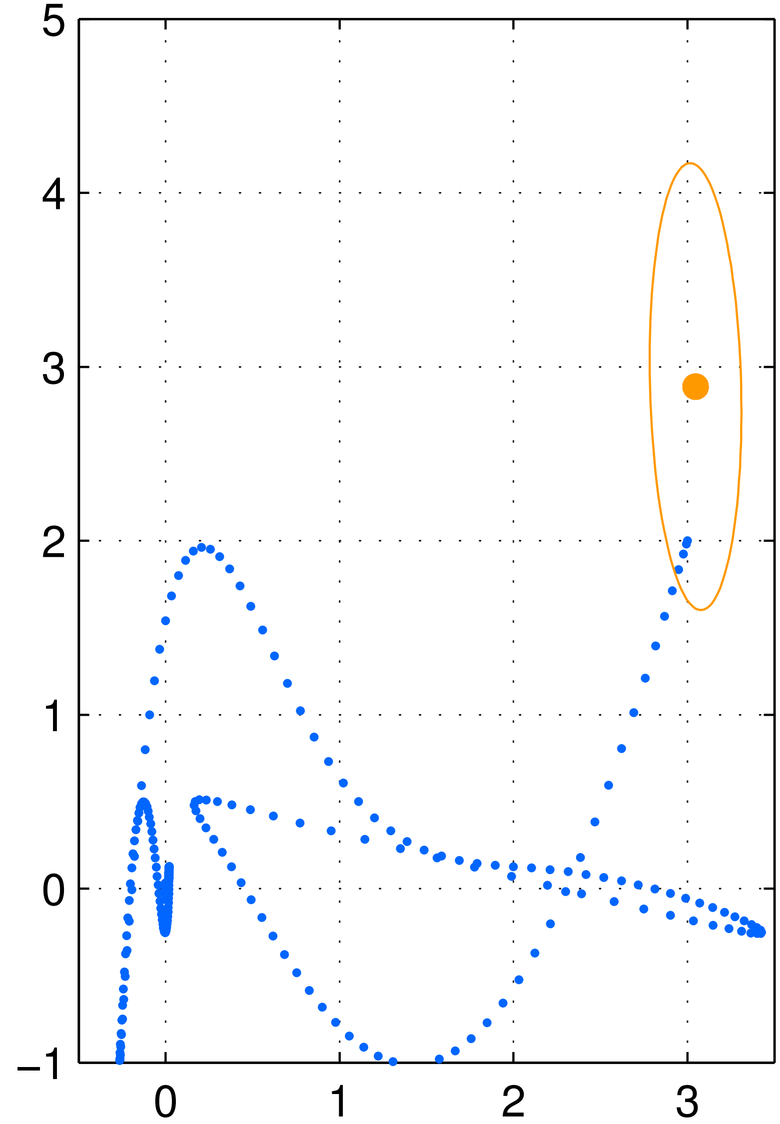


example: estimate and error ellipsoids

after 3 measurements and after 20 measurements

estimate and worst-case ellipsoid with unit norm noise





estimating other states

we can also use least squares to estimate later states $x(t)$

once we have an estimate of $x(0)$, then

$$x_{\text{est}}(t) = A^t x_{\text{est}}(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} B u(\tau)$$

we'll analyze the error properties in Engr207b

example: estimating other states

Estimating all states based on a 20-measurement estimate of $x(0)$.