5. State-Space Systems

- Representing systems as first-order ODES
- Systems as maps
- Controllability and observability
- The order of a realization
- Minimal realizations
- Matrix-valued transfer functions
- Realizations for matrix transfer functions
This form is often called state-space form.

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]

where

- \(x(t) \in \mathbb{R}^n\) is called the state.
- \(u(t) \in \mathbb{R}^m\) is called the input signal or forcing function.
- \(A \in \mathbb{R}^{n \times n}\) is the generator or dynamics matrix.
- \(B \in \mathbb{R}^{n \times m}\) is the forcing function.

System of differential equations

Linear first-order ODEs
Mechanical system with \( k \) degrees of freedom undergoing small motions:

\[
\ddot{q}(t) + \dot{\dot{q}}(t) + \dddot{q}(t) = F(t)
\]

where
- \( q(t) \in \mathbb{R}^k \) represents the configuration or generalized coordinates of the system.
- \( M \) is the mass matrix.
- \( D \) is the damping matrix.
- \( K \) is the stiffness matrix.
- \( W \) is the mass matrix.
- \( f(t) \) represents the configuration or generalized coordinates of the system.

State-space form

Let the state be \( x(t) = \begin{bmatrix} q(t) & \dot{q}(t) \end{bmatrix}^T \),

\[
\dot{x}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f(t)
\]

Where

\[
(\dot{t})A = (\dot{t})bX + (\dot{t})bD + (\dot{t})bW
\]
Autonomous behavior

System behavior when \( u(t) = 0 \) for all \( t \).

The solution is given by

\[
(0x)\Phi = (\dot{t})x
\]

\( \Phi \) is called the state transition matrix. The map \( \Phi \) is linear, hence we can represent it as a matrix.

\( \Phi \) maps initial state to state at time \( t \).

Notes

- \( \Phi \) maps \( \mathbb{R}^n \) to \( \mathbb{R}^n \).
- \( \Phi \) is linear. Hence, we can represent it as a matrix.

System behavior when \( u = 0 \) for all \( t \).
The state transition matrix is \( \Phi_t = e^{At} \) where the matrix exponential is
\[
e^M = I + M + \frac{M^2}{2} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots
\]
This series always converges.

Properties

- \( e^M \) is invertible.
- \( e^0 = I \) for the zero matrix.
- \((e^M)^* = (e^M)^* \)
- If \( M \) and \( N \) are square, then \( e^M e^N = e^{M+N} \)

Autonomous behavior

\[ W = N W \iff N^\vartheta W^\vartheta = N + W^\vartheta \]

where the matrix exponential is
\[ W^\vartheta = \Phi \]

The state transition matrix is
The system \( \dot{x}(t) = Ax(t) \) is called **internally stable** if \( x(t) \to 0 \) as \( t \to \infty \) for every initial condition \( x(0) \).

**Theorem**

The system is internally stable if and only if all of the eigenvalues of \( A \) have strictly negative real part. That is, if \( \Re(\lambda) < 0 \) for all \( \lambda \in \text{spec}(A) \).

The **spectrum** of a matrix is the set of its eigenvalues:

\[
\text{spec}(A) = \{ \lambda \in \mathbb{C} ; \lambda I - A \text{ is singular} \}
\]

For every initial condition \( x(0) \),

\[
\infty \leftarrow t \quad \text{as} \quad 0 \leftarrow (t)x
\]

is called internally stable if \( (t)x \forall t \) is.

The system **stability**
Reachable States

The set of reachable states at time $t > 0$ is

$$\begin{align*}
\mathcal{R}_t &= \{ \xi \in \mathbb{R}^n \mid \text{there exists } u \in \mathbb{R}^m \text{ such that } x(t) = \xi \} \\
\end{align*}$$

Properties

$\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$.

$\mathcal{R}_t$ is independent of time $t > 0$.

$\mathcal{R}_t$ is called the controllable subspace.

$\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$.

The system is called controllable if $\mathcal{R}_t = \mathbb{R}^n$. 

Follows from linearity.

Reachable States
The controllability matrix is the image of the controllability matrix.

The controllable subspace is the image of the controllability matrix.

\[
\mathcal{R}^c = \text{image}(\mathcal{C})
\]

Main Theorem

\[
\begin{bmatrix}
\mathcal{C} \mathcal{A} \mathcal{B} & \mathcal{A} \mathcal{B} & \cdots & \mathcal{A}^{n-1} \mathcal{B}
\end{bmatrix} = \mathcal{C} \mathcal{A} \mathcal{B}
\]

The controllability matrix is

Controllability
Systems with inputs and outputs

General system form

\[ \dot{x}(t) = A x(t) + B u(t) \]
with initial condition \( x(0) = 0 \)

\[ y(t) = C x(t) + D u(t) \]

Here \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^p \).

System \( G \) is a black box mapping signals to signals \( y \).

Standard interpretation

\[ u(t) \in \mathbb{R}^m \text{ and } y(t) \in \mathbb{R}^p \]

Here \( x(0) \in \mathbb{R}^n \), and \( (t) \in \mathbb{R} \).

with initial condition \( x \)

\[ (t) d(t) + (t) x(t) = (t) y(t) \]

General system form

Systems with inputs and outputs

Write \( G : \mathcal{X} \rightarrow \mathcal{Y} \) is a \( (t) \) map.

If \( x(0) = 0 \) then \( G \) is a linear map.

Draw diagram with blocks and arrows.
We will see more on this later.

This has no state-space form.

\[(q)\dot{n} = (q)\ddot{u}\]

Not every system can be represented in state-space form. E.g.

Caveat

State-space form

\[
\begin{bmatrix}
0 \\
I \\
0 \\
\vdots \\
0
\end{bmatrix} = B
\]

\[
\begin{bmatrix}
1^{-u} I & \cdots & 1 & 0
\end{bmatrix} = C
\]

\[
\begin{bmatrix}
1^{-u} I & \cdots & 1 & 0
\end{bmatrix} = A
\]

General systems of ODEs

\[
n_0 C + n_1 C + \cdots + 1^{-u} n_{1-u} C = \ddot{h}_0 p + \ddot{h}_1 u + \cdots + (1-u) \ddot{h}_{1-u} p + (u) \ddot{h}
\]
There is a unique solution for $x_0$ if and only if $\ker(\Phi) = \{0\}$

$$n^T V - \Phi = 0 x_0$$

To find $x_0$ we need to solve the equation.

Given $y$ and $u$, can we uniquely determine $x$?

The question of observability

The map on signals $y$ and $u$, we have

$$\Psi_t x_0 = y - \Lambda_t u$$

Here $\Psi_t$ and $\Lambda_t$ are linear maps.

As a map on signals $y$ and $u$, we have

$$\Psi_t x_0 = y - \Lambda_t u$$

The solution is

$$0 x = (0) x$$

with initial condition

$$x = x$$

General system form

Observability
Observability

The set of unobservable states at time $t > 0$ is

\[ \mathcal{U}_t = \ker(\Psi_t) = \{ \xi \in \mathbb{R}^n ; \Psi_t \xi = 0 \} \]

- $\mathcal{U}_t$ is a subspace of $\mathbb{R}^n$.

- If $\xi \in \mathcal{U}_t$, then the initial condition $x_0$ and the initial condition $x_0 + \xi$ will produce the same output on $[0, t]$ for every $u$.

- If $\xi \not\in \mathcal{U}_t$, then $\mathcal{U}_t$ is a subspace of $\mathbb{R}^n$.

The observability matrix is

\[ \begin{bmatrix} u \mathcal{O} & \vdots & \mathcal{O} \\ \vdots & \ddots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} \end{bmatrix} = \mathcal{O} \quad \text{where} \quad \ker(\mathcal{O}) = \mathcal{U}_t \]

Facts

- $\mathcal{U}_t = \ker(\mathcal{O} \mathcal{C} \mathcal{A})$ where $\mathcal{O} \mathcal{C} \mathcal{A} = \begin{bmatrix} \mathcal{C} \\ \mathcal{C} \mathcal{A} \\ \vdots \\ \mathcal{C} \mathcal{A}^{n-1} \end{bmatrix}$, the observability matrix.

- Write $\mathcal{N} \mathcal{C} \mathcal{A} = \ker(\mathcal{O} \mathcal{C} \mathcal{A})$.

- $\mathcal{N}$ is independent of time.

If $\mathcal{O} \mathcal{C} \mathcal{A}$ has full rank, then the system is called observable.

\[ \begin{bmatrix} u \mathcal{O} & \vdots & \mathcal{O} \\ \vdots & \ddots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} \end{bmatrix} = \mathcal{O} \quad \text{where} \quad \ker(\mathcal{O}) = \mathcal{U}_t = \mathcal{N} \]
Suppose $G_1$ and $G_2$ are state-space systems, with zero initial conditions. $G_1$ and $G_2$ are called equivalent if
State coordinate changes

Let $G$ be the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Let $z(t) = T x(t)$ for some invertible matrix $T \in \mathbb{R}^{n \times n}$. Then

$$\dot{z}(t) = TAT^{-1}z(t) + T Bu(t)$$

$$y(t) = CT^{-1}z(t) + Du(t)$$

Let $C$ be the system

State coordinate changes

$$(t)\dot{z} + (t)z = (t)\dot{h}$$

$$(t)z + (t)x = (t)h$$

$$(t)\dot{x} + (t)x = (t)x$$
State coordinate changes

\[
\begin{align*}
(\mathcal{T})n + (\mathcal{T})z \begin{bmatrix} z & I^\bot \end{bmatrix} &= (\mathcal{T})\hat{y} \\
(\mathcal{T})n \begin{bmatrix} z \\ I \end{bmatrix} + (\mathcal{T})z \begin{bmatrix} z & 0 \\ 0 & I^\bot \end{bmatrix} &= (\mathcal{T})\hat{z}
\end{align*}
\]

we can represent the same map from \( u \) to \( y \) by

\[
(\mathcal{T})x \begin{bmatrix} I & I \\ z & I \end{bmatrix} = (\mathcal{T})xL = (\mathcal{T})z
\]

Changing coordinates to

\[
(\mathcal{T})n + (\mathcal{T})x \begin{bmatrix} 0 & I \end{bmatrix} = (\mathcal{T})\hat{y}
\]

\[
(\mathcal{T})n \begin{bmatrix} 0 \\ I \end{bmatrix} + (\mathcal{T})x \begin{bmatrix} z & z^\bot \\ I & 0 \end{bmatrix} = (\mathcal{T})\hat{x}
\]

Example

\[
\begin{align*}
(\mathcal{T})n + (\mathcal{T})z \begin{bmatrix} z & I^\bot \end{bmatrix} &= (\mathcal{T})\hat{y} \\
(\mathcal{T})n \begin{bmatrix} z \\ I \end{bmatrix} + (\mathcal{T})z \begin{bmatrix} z & 0 \\ 0 & I^\bot \end{bmatrix} &= (\mathcal{T})\hat{z}
\end{align*}
\]

\[
(\mathcal{T})x \begin{bmatrix} I & I \\ z & I \end{bmatrix} = (\mathcal{T})xL = (\mathcal{T})z
\]

Changing coordinates to

\[
(\mathcal{T})n + (\mathcal{T})x \begin{bmatrix} 0 & I \end{bmatrix} = (\mathcal{T})\hat{y}
\]

\[
(\mathcal{T})n \begin{bmatrix} 0 \\ I \end{bmatrix} + (\mathcal{T})x \begin{bmatrix} z & z^\bot \\ I & 0 \end{bmatrix} = (\mathcal{T})\hat{x}
\]

Controllability and observability are preserved under state coordinate changes. That is, \( \text{rank}(CAB) \) \( \text{rank}(COA) \) are unchanged.

Transforms from one realization for \( G \) to another.

\[
(\mathcal{T})n \begin{bmatrix} A & B, C^\top L, D \end{bmatrix} \leftrightarrow (\mathcal{T})n \begin{bmatrix} A', B', C^\top L', D' \end{bmatrix}
\]
System equivalence

When are two systems equivalent?

Theorem: Suppose \((A_1, B_1, C_1, D_1)\) and \((A_2, B_2, C_2, D_2)\) are realizations for \(G_1\) and \(G_2\) respectively. Then when are two systems are equivalent?

System equivalence

Proof

We have, for any realization \((A', B', C', D')\)

\[
\begin{align*}
(0)^n A + \frac{\partial}{\partial t} n B e^{(t-\tau)A} C & = \int_{0}^{t} \frac{\partial}{\partial \tau} \int_{0}^{\tau} \frac{\partial}{\partial \tau} B e^{(\tau-\tau')A} C d\tau' d\tau \\
& = (t)^n A
\end{align*}
\]

for all \(t\), since \(D_1^n u(0) = D_2^n u(0)\) for all functions \(u\) and for all \(t\).

We need to show that

\[
(0)^n A = (0)^n B e^{A t} C - C_2 e^{A_2 t} B_2
\]

for all \(t\) and for all \(n\) and for all \(t\)

\[
C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2
\]

\[
\begin{align*}
\Leftrightarrow & \quad C_1 \text{ and } C_2 \text{ are equivalent} \\
\Leftrightarrow & \quad G_1 \text{ and } G_2 \text{ are equivalent for all } t
\end{align*}
\]

\[
\begin{align*}
\Leftrightarrow & \quad G_1 \text{ and } G_2 \text{ are equivalent for all } t
\end{align*}
\]

for all \(t\) and for all \(n\) and for all \(t\)

\[
\begin{align*}
\Leftrightarrow & \quad C_1 \text{ and } C_2 \text{ are equivalent} \\
\Leftrightarrow & \quad G_1 \text{ and } G_2 \text{ are equivalent for all } t
\end{align*}
\]

for all \(t\) and for all \(n\) and for all \(t\)

\[
\begin{align*}
\Leftrightarrow & \quad C_1 \text{ and } C_2 \text{ are equivalent} \\
\Leftrightarrow & \quad G_1 \text{ and } G_2 \text{ are equivalent for all } t
\end{align*}
\]
The proof in the matrix valued case is similar, which contradicts our assumption that the above integral is zero.

\[ 0 < \lim_{\tau \to 0} |(\tau)n| \int_{\tau-\tau}^{\tau+\tau} = \lim_{\tau \to 0} (\tau)n(\tau - I + 0\tau)A \int_{\tau-\tau}^{\tau+\tau} \]

and choose \( 0 \neq (I)n \). This gives \( n = I + 0\tau \).

To show a contradiction, assume the above integral is zero for all \( t \) and \( u \), yet there is some \( t_0 \geq 0 \) for which \( F(t_0) \neq 0 \). Pick \( u(t) = F(t_0 + 1 - t) \) and choose \( t = t_0 + 1 \). This gives

\[ \int_{t_0}^{t_0+1} \int_{t_0}^{t_0+1} F(t_0+1-\tau)u(\tau)d\tau = \int_{t_0}^{t_0+1} |u(\tau)|^2 d\tau > 0 \]

which contradicts our assumption that the above integral is zero.

The proof in the matrix valued case is similar.
Removing uncontrollable states

The dynamic order or state-dimension of a state-space system is the dimension \( n \) of the generator matrix \( A \).

If a system is not controllable, then there exists an equivalent lower-order realization.

**Theorem:** If \( \dim(C) = r \), then there exists an equivalent lower-order realization.

\[
\begin{bmatrix}
0 \\
\mathbf{B}
\end{bmatrix} \begin{bmatrix}
\mathbf{I} & 0 \\
\mathbf{I} & \mathbf{A}
\end{bmatrix} \begin{bmatrix}
\mathbf{C}_1 \\
\mathbf{C}_2
\end{bmatrix} = \mathbf{B} \mathbf{A}^r \mathbf{C}
\]

Equivalence follows from the representation, because

This representation is called controllability form.

Notes

1. The lower-order system \((\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)\) is equivalent to \((A, B, C, D)\), and is controllable.

\[
\begin{bmatrix}
0 \\
\mathbf{B}
\end{bmatrix} = \mathbf{B} \mathbf{I} = \mathbf{B}
\]

\[
\begin{bmatrix}
\mathbf{C}_1 \\
\mathbf{C}_2
\end{bmatrix} = \mathbf{I} \mathbf{C} = \mathbf{C}
\]

\[
\begin{bmatrix}
\mathbf{A} \\
\mathbf{0}
\end{bmatrix} \begin{bmatrix}
\mathbf{A} \\
\mathbf{I}
\end{bmatrix} = \mathbf{I} \mathbf{A} \mathbf{L} = \mathbf{A}
\]

Theorem: If \( \dim(C) = r \), then we can choose coordinates so that

A system is not controllable, then there exists an equivalent lower-order realization.

Generator matrix \( A \).

The dynamic order or state-dimension of a state-space system is the dimension of the state-space states.
Removing uncontrollable states

Example

The state component $x_2$ is uncontrollable. With initial condition $x(0) = 0$, the state component $x_2(0) = 0$ for all $t$. The second-order state-space system

$$\dot{x}(t) = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 10 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} x(t)$$

represents the same map as the first-order system

$$\dot{z}(t) = -z(t) + u(t)$$

$$y(t) = -x(t)$$

The state component $x_2$ is uncontrollable. With initial condition $x(0) = 0$, the state component $x_2(t) = 0$ for all $t$.

The second-order state-space system

$$(\dot{t})x = \begin{bmatrix} 0 & 1 \\ -1 \end{bmatrix} = (\dot{t})\hat{x}$$

$$(\dot{t})n + (\dot{t})z = (\dot{t})\hat{z}$$

represents the same map as the first-order system

$$(\dot{t})x \begin{bmatrix} 0 & 1 \\ -1 \end{bmatrix} = (\dot{t})\hat{x}$$

$$(\dot{t})n \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (\dot{t})x \begin{bmatrix} 0 & -1 \\ -1 \end{bmatrix} = (\dot{t})x$$

The second-order state-space system

$$(\dot{t})x \begin{bmatrix} 0 & 1 \\ -1 \end{bmatrix} = (\dot{t})\hat{x}$$

$$(\dot{t})n \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (\dot{t})x \begin{bmatrix} 0 & -1 \\ -1 \end{bmatrix} = (\dot{t})x$$

removing uncontrollable states
We first show that the controllable subspace is $A$-invariant. 

$x \in C_{AB} \Rightarrow Ax \in C_{AB}$

This holds because, if $x \in C_{AB}$, then $Ax \in C_{AB}$.
Now choose coordinates $x_L = z$ such that

$$\left\{ 0 = z^T \in \mathbb{R}^n, \exists \in C \right\} = \mathbb{R}^m C$$

Proof continued
The ideas of controllability and observability are called dual.

**Duality**

As for controllability, noting that the unobservable subspace is A-invariant.

**Proof**

This representation is called observability form.

The lower-order system \((A_{11}, B_1, C_1, D_1)\) is equivalent to \((A, B, C, D)\), and is observable.

\[
\begin{bmatrix}
A_{11} \\
B_1
\end{bmatrix} = B
\]

\[
\begin{bmatrix}
0 & I_{r} \\
\nu & A
\end{bmatrix} = \mathcal{A}
\]

If \(\dim(\mathcal{N}(A)) = n - r\), then we can choose coordinates so that removing unobservable states...
with the last equality following from the previous one at $t = 0$.

For all $k$

$$C^1 A^k V_1 B = C^2 A^k V_2 B$$

$\Leftarrow$

For all $t$ and $k$

$$C^1 A^k e^{A^1 t} B = C^2 A^k e^{A^2 t} B$$

$\Leftarrow$

For all $t$ and $k$

$$C^1 e^{A^1 t} A^k V_1 B = C^2 e^{A^2 t} A^k V_2 B$$

$\Leftarrow$

For all $t$

$$C^1 e^{A^1 t} V_1 B = C^2 e^{A^2 t} V_2 B$$

$\Leftarrow$

For the direction, we know

$$\cdots + \frac{1}{2} C^1 A^2 V_1 B + C A V_1 B + C B = C^2 A^2 V_2 B + C A V_2 B + C B$$

Proof: The direction follows immediately from the previous Lemma, since the matrices $C^1 A B, C A V_1 B, \ldots$ are called the Markov parameters for $G$.

For all $k$

$$D^1 = D^2$$

$\Leftarrow$\qquad $G^1$ and $G^2$ are equivalent

Then $G^2$ respectively. Then

Theorem: Suppose that $V_1 B, I, C^1, C^2, D^1$ and $V_2 B, I, C^2, C^1, D^2$ are realizations for $G^1$ and $G^2$, respectively. Then another characterization of equivalence
The minimal n for which a realization exists is a property of the map G.

- We will use the equality of the Markov parameters to prove the $\Rightarrow$ direction.
- We have already shown the $\Leftarrow$ direction.

**Notes**

- We have already shown the $\Rightarrow$ direction.
- We will use the equality of the Markov parameters to prove the $\Leftarrow$ direction.
- The minimum n for which a realization exists is a property of the map G.

**Theorem:** A realization for G with smaller state dimension.

A realization (A', B', C', D') for a system G is called minimal if there does not exist a minimal realizations.

\[ (A', B', C', D') \text{ is minimal if and only if } \forall (A, B, C) \text{ is observable and } (A, C) \text{ is controllable} \]
Proof

We need to show the direction. Suppose \((A^r, B^r, C^r, D^r)\) is controllable and observable, then it is an equivalent realization. We will show that if \((A^1, B^1, C^1, D^1)\) is an equivalent realization, then it must have order at least \(n\). Hence \(\text{rank}(CA_1A^1B^1)\) has at least \(u\) rows, and therefore \(A^1\) has at least \(n\) columns. Hence \(\text{rank}(CA_1A^1B^1)\geq u\), which implies that \(\text{rank}(OCA_1A^1B^1)\geq u\), from the left Sylvester inequality.

\[
\begin{bmatrix}
\begin{bmatrix}
OCA_1A^1B^1 \\
\vdots \\
OCA_1A^1B^1\\
C_1A^1B^1 \\
C_1A^1B^1
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
[\begin{bmatrix}
A^1 & \cdots & A^1 \\
B^1 & A^1 & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
C_1 & \cdots & C_1
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
A^1 & \cdots & A^1 \\
B^1 & A^1 & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
C & \cdots & C
\end{bmatrix}
\]

For any two matrices \(P\) and \(Q\), we have Sylvester's inequality:

\[
\text{rank}(PQ) \leq \min\{\text{rank}(P), \text{rank}(Q)\}
\]

We know that \(\text{rank}(OCA_1A^1B^1)\geq u\), from the left Sylvester inequality. Hence \(\text{rank}(OCA_1A^1B^1)\geq u\), which implies that \(\text{rank}(OCA_1A^1B^1)\geq n\) and \(\text{rank}(CA_1A^1B^1)\geq n\) from the right Sylvester inequality. Hence \(\text{rank}(OCA_1A^1B^1)\geq u\), from the left Sylvester inequality.

\[
\begin{bmatrix}
(\partial)\{\text{rank}\} \geq (\partial)\text{rank} \geq u - (\partial)\text{rank} + (\partial)\text{rank}
\end{bmatrix}
\]

We need to show the direction. Suppose \((A^r, B^r, C^r, D^r)\) is controllable and observable, then it is an equivalent realization. We will show that if \((A^1, B^1, C^1, D^1)\) is an equivalent realization, then it must have order at least \(n\). Hence \(\text{rank}(CA_1A^1B^1)\) has at least \(u\) rows, and therefore \(A^1\) has at least \(n\) columns. Hence \(\text{rank}(CA_1A^1B^1)\geq u\), which implies that \(\text{rank}(OCA_1A^1B^1)\geq u\), from the left Sylvester inequality.

\[
\begin{bmatrix}
\begin{bmatrix}
OCA_1A^1B^1 \\
\vdots \\
OCA_1A^1B^1\\
C_1A^1B^1 \\
C_1A^1B^1
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
[\begin{bmatrix}
A^1 & \cdots & A^1 \\
B^1 & A^1 & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
C_1 & \cdots & C_1
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
A^1 & \cdots & A^1 \\
B^1 & A^1 & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
C & \cdots & C
\end{bmatrix}
\]

We need to show the direction. Suppose \((A^r, B^r, C^r, D^r)\) is controllable and observable, then it is an equivalent realization. We will show that if \((A^1, B^1, C^1, D^1)\) is an equivalent realization, then it must have order at least \(n\). Hence \(\text{rank}(CA_1A^1B^1)\) has at least \(u\) rows, and therefore \(A^1\) has at least \(n\) columns. Hence \(\text{rank}(CA_1A^1B^1)\geq u\), which implies that \(\text{rank}(OCA_1A^1B^1)\geq u\), from the left Sylvester inequality.

\[
\begin{bmatrix}
(\partial)\{\text{rank}\} \geq (\partial)\text{rank} \geq u - (\partial)\text{rank} + (\partial)\text{rank}
\end{bmatrix}
\]
\[ A + B_{1-}(V - Is)C = (s) \begin{bmatrix} D \\ B \end{bmatrix} \]

Transfer functions

Recall the Laplace transform of \( f(t) \):

\[ \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \]

- The Laplace transform is a linear map.
- \( \mathcal{L}\{f(t)\} \) has a Laplace transform, then it is given by \( \mathcal{L}\{f(t)\} \).
- Applying the Laplace transform to the initial condition \( x(0) = 0 \):

\[ (s)f + (s)x = (s)\hat{y} \]

\[ (s)f + (s)x = (s)x \]

\[ (s)f = (s)x \]

\[ \mathcal{L}\{f(t)\} = (s)f \]

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Transfer functions

The function $G: \mathbb{C}^m \rightarrow \mathbb{C}^p$ is called the transfer function:

$$G(s) = \frac{C(sI - A)^{-1}B + D}{0q + s^1q + \cdots + s^{n-1}q + 1 - ws^nq} = (s)b$$

It is called real-rational if the coefficients are real. It is called proper if $n \geq m$, and strictly proper if $n < m$.

Rational functions

• A scalar function $g: \mathbb{C} \rightarrow \mathbb{C}$ is called rational if

$$\frac{0q + \cdots + 1 - ws^nq}{0q + s^1q + \cdots + 1 - ws^nq} = (s)b$$

• If $n \geq m$, the function is called proper.

Transfer functions

$D + B_{1-VI} = (s)c$
We call \( \hat{G} \) proper if each of its entries is proper.

\[
\forall \quad A - Is^{-1} = \left[ \text{cofactor of element } ij \right] \times \frac{(A - Is)^{\text{det}}}{I} = \left[ 1 - (A - Is) \right]
\]

where each cofactor is the determinant of a submatrix of \( A - Is \).

The function \( \hat{G} \) corresponding to a state-space system is rational, since we call the matrix-valued function \( \hat{G} \) rational if each of its entries is rational.
Given $G^1$ and $G^2$ defined by state-space representations $(A^1, B^1, C^1, D^1)$ and $(A^2, B^2, C^2, D^2)$ respectively, given $G^1$ and $G^2$ are defined by state-space representations $(A^1, B^1, C^1, D^1)$ and $(A^2, B^2, C^2, D^2)$ respectively,

The if part follows by equality as $s \to \infty$.

\[ \mathcal{L} \{ A^1 - (A^2 - IS^2)C^1 \} = A^1 - (A^2 - IS^2)C^1 \]

which holds if and only if

\[ \mathcal{L} \{ A^1 - (A^2 - IS^2)C^1 \} = A^1 - (A^2 - IS^2)C^1 \]

since the Laplace transform of $e^{A^1 t}$ is $(sI - A^1)^{-1}$, this is equivalent to

\[ \mathcal{L} \{ A^1 - (A^2 - IS^2)C^1 \} = A^1 - (A^2 - IS^2)C^1 \]

We know $G^1$ and $G^2$ are equivalent
Given a scalar-valued (often called SISO) strictly proper transfer function $\hat{g}(s)$, there exists a state-space realization $(A, B, C, D)$ which has order $n$.

$$0 = A \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} = B$$

$$\begin{bmatrix} 1 - u_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 - u_{n-1} \\ 0 & \cdots & 1 \end{bmatrix} = C$$

$$\begin{bmatrix} 1 - u_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = A$$

### Proof

It is...
If \( \hat{g} \) is proper but not strictly proper, we can write it as

\[
\hat{g}(s) = \hat{g}_1(s) + D(s)
\]

where \( \hat{g}_1 \) is strictly proper.

\[
A + (s)\hat{b} = (s)\hat{b}
\]

If \( \hat{b} \) is proper but not strictly proper, we can write it as non-strictly proper.
To realize a matrix-valued transfer function $\tilde{G}$, we can do so in blocks.

Suppose we have realizations $$(A_1, B_1, C_1, D_1)$$ for $\tilde{G}_1$ and $$(A_2, B_2, C_2, D_2)$$ for $\tilde{G}_2$.

Then a realization for $G$ is

$$\begin{bmatrix}
\begin{array}{c|c}
\tilde{G}_1 & 0 \\
\hline
0 & \tilde{G}_2
\end{array}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c|c}
A_1 & 0 \\
\hline
0 & A_2
\end{array}
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{c|c}
(s)\tilde{G}_1 & (s)\tilde{G}_2 \\
\hline
(s)C_1 & (s)C_2
\end{array}
\end{bmatrix}
$$
Suppose \( \hat{G}(s) = \begin{bmatrix} \hat{G}_1(s) & \hat{G}_2(s) \end{bmatrix} \). Then a realization for \( G \) is

\[
\begin{bmatrix}
A_2 & 0 & 0 & 0 \\
B_2 & A_1 & 0 & 0 \\
B_1 & 0 & A_1 & 0 \\
0 & 0 & 0 & D_1
\end{bmatrix}
\begin{bmatrix}
(s) \hat{G}_1 \\
(s) \hat{G}_2 \\
(s) \hat{G}_1 \\
(s) \hat{G}_2 \\
\end{bmatrix}
= (s) \hat{G}
\]
A procedure for realization of a rational transfer matrix \( \hat{G} \) is

1. Realize each element \( \hat{G}_{ij} \), which is a scalar transfer function.
2. Realize the columns.
3. Realize the row of columns.

Caveat

The resulting realization may be non-minimal. For example,

\[
\begin{bmatrix}
0 & 0 & I \\
\frac{s}{2} & I & 0 \\
I & 0 & -s \\
0 & I & 0
\end{bmatrix} = (s) \hat{G}
\]

The previous construction leads to

\[
\begin{bmatrix}
I + \frac{s}{2} & I + s \\
I & 2
\end{bmatrix} = (s) \hat{G}
\]

but a lower-order realization is

\[
\begin{bmatrix}
0 & 0 & I \\
\frac{s}{2} & I & 0 \\
I & 0 & -s \\
0 & I & 0
\end{bmatrix} = (s) \hat{G}
\]
We need a notion of approximation for systems. More later...

\[
\begin{bmatrix}
0.0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
= 0 \\
1
\end{bmatrix}
\]

which has singular values \( \sigma \)

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= p A B = p A C
\]

\[
(\tau) x \begin{bmatrix} 0 & 1 \end{bmatrix} = (\tau) \dot{x}
\]

\[
(\tau) n \begin{bmatrix} 0 & 1 \end{bmatrix} + (\tau) x \begin{bmatrix} 2 & 1 \\
3 & -1 \end{bmatrix} = (\tau) x
\]

It can go wrong in similar ways; e.g.

Platonic theory of systems

Analogous to the idea of rank of a matrix, we have the notion of order of a linear system.

View systems as linear operators on signal spaces. The map between inputs and outputs defines the system.

Every state-space system has a proper transfer function representation.

Every proper rational transfer matrix has a state-space realization.