5. State-Space Systems

- Representing systems as first-order ODEs
- Systems as maps
- Controllability and observability
- The order of a realization
- Minimal realizations
- Matrix-valued transfer functions
- Realizations for matrix transfer-functions

Linear first-order ODEs

System of differential equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

- $x(t) \in \mathbb{R}^n$ is called the *state*.
- $u(t) \in \mathbb{R}^m$ is called the *input signal* or *forcing function*.
- $A \in \mathbb{R}^{n \times n}$ is the *generator* or *dynamics matrix*.
- $B \in \mathbb{R}^{n \times m}$.

This form is often called *state-space* form.

Mechanical systems

Mechanical system with k degrees of freedom undergoing small motions

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = F(t)$$

where

- $q(t) \in \mathbb{R}^k$ represents the *configuration* or *generalized coordinates* of the system.
- M is the *mass matrix*.
- K is the *stiffness matrix*.
- D is the *damping matrix*.

State-space form

Let the state be $x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$.

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} F(t)$$

Autonomous behavior

System behavior when $u(t) = 0$ for all t.

$$\dot{x}(t) = Ax(t) \qquad \text{with initial condition } x(0) = x_0$$

The solution is given by

$$x(t) = \Phi_t(x_0)$$

Notes

- $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ maps initial state to state at time t.
- The map Φ_t is linear; hence we can represent it as a matrix.
- Φ_t is called the *state transition matrix*.

Autonomous behavior

The state transition matrix is

$$\Phi_t = e^{At}$$

where the *matrix exponential* is

$$e^M = I + M + \frac{M^2}{2} + \frac{M^3}{3!} + \frac{M^4}{4!} + \dots$$

This series always converges.

Properties

- e^M is invertible.
- $e^0 = I$ for the zero matrix.
- $e^{M^*} = (e^M)^*$
- $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
- If M and N are square, then

$$e^{M+N} = e^M e^N \qquad \iff \qquad MN = NM$$

Stability

The system

$$\dot{x}(t) = Ax(t)$$

is called *internally stable* if

$$x(t) \to 0 \qquad \text{as } t \to \infty$$

for every initial condition $x(0)$.

Theorem

The system is internally stable if and only all of the eigenvalues of A have strictly negative real part. That is, if

$$\mathrm{Re}(\lambda) < 0 \text{ for all } \lambda \in \mathrm{spec}(A)$$

The *spectrum* of a matrix is the set of its eigenvalues

$$\mathrm{spec}(A) = \Big\{ \lambda \in \mathbb{C} \;;\; \lambda I - A \text{ is singular} \Big\}$$

Reachable States

The set of *reachable states* at time $t > 0$ is

$$\mathcal{R}_t = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \text{there exists } u \text{ such that } x(t) = \xi \right\}$$

Properties

- $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$.

  Follows from linearity.

- $\mathcal{R}_t$ is independent of time $t > 0$.

- $\mathcal{R}_t$ is called the *controllable subspace*.

- The system is called *controllable* if $\mathcal{R}_t = \mathbb{R}^n$.

Controllability

The *controllability matrix* is

$$C_{AB} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Main Theorem

The controllable subspace is the image of the controllability matrix

$$\mathcal{R}_t = \operatorname{image}(C_{AB})$$

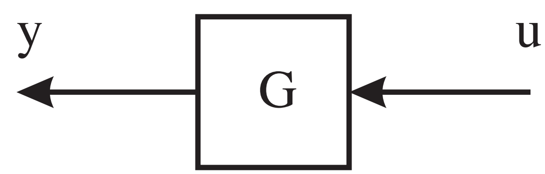
Systems with inputs and outputs

General system form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{with initial condition } x(0) = 0$$
$$y(t) = Cx(t) + Du(t)$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

Standard interpretation



- System G is a 'black box' mapping signals u to signals y.
- If $x(0) = 0$ then G is a linear map.
- Write $G : \mathcal{F} \rightarrow \mathcal{F}$, and $y = Gu$. Function spaces to be defined later.

General systems of ODEs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = c_{n-1}u^{n-1} + \cdots + c_1\dot{u} + c_0 u$$

State-space form

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix} \qquad D = 0$$

Caveat

Not every system can be represented in state-space form. e.g.

$$y(t) = \dot{u}(t)$$

has no state-space form.

We will see more on this later.

Observability

General system form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{with initial condition } x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

The solution is

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

As a map on signals y and u, we have

$$y = \Psi_t x_0 + \Lambda_t u$$

Here $\Psi_t : \mathbb{R}^n \to \mathcal{F}([0,t],\mathbb{R}^p)$ and $\Lambda_t : \mathcal{F}([0,t],\mathbb{R}^m) \to \mathcal{F}([0,t],\mathbb{R}^p)$ are linear maps.

The question of observability

Given y and u, can we uniquely determine x_0?
To find x_0 we need to solve the equation

$$\Psi_t x_0 = y - \Lambda_t u$$

There is a unique solution for x_0 if and only if $\ker(\Psi_t) = \{0\}$.

Observability

The set of *unobservable states* at time $t > 0$ is

$$\begin{aligned} \mathcal{U}_t &= \ker(\Psi_t) \\ &= \left\{ \xi \in \mathbb{R}^n \; ; \; \Psi_t \xi = 0 \right\} \end{aligned}$$

- $\mathcal{U}_t$ is a subspace of $\mathbb{R}^n$.
- If $\xi \in \mathcal{U}_t$, then the initial condition x_0 and the initial condition $x_0 + \xi$ will produce the same output on $[0, t]$ for every u.

Facts

- $\mathcal{U}_t = \ker(O_{CA})$ where $O_{CA} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$, the *observability matrix*.
- Write $\mathcal{N}_{\mathcal{CA}} = \ker(O_{CA})$.
- $\mathcal{U}_t$ is independent of time.
- If $\operatorname{rank}(O_{CA}) = n$ then the system is called *observable*.

Systems as maps

Suppose G_1 and G_2 are state-space systems, with zero initial conditions. G_1 and G_2 are called *equivalent* if

$$G_1 u = G_2 u \qquad \text{for all inputs } u$$

Notes

- Given a map G, there are many sets of matrices (A, B, C, D) which result in the same map.
- Any particular set of matrices (A, B, C, D) which represent G is called a *realization* for G.

State coordinate changes

Let G be the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Let $z(t) = Tx(t)$ for some invertible matrix $T \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned}\dot{z}(t) &= TAT^{-1}z(t) + TBu(t)\\ y(t) &= CT^{-1}z(t) + Du(t)\end{aligned}$$

State coordinate changes

Mapping

$$(A, B, C, D) \qquad \mapsto \qquad (TAT^{-1}, TB, CT^{-1}, D)$$

transforms from one realization for G to another.

Controllability and observability are preserved under state coordinate changes. That is, $\operatorname{rank}(C_{AB})$ and $\operatorname{rank}(O_{CA})$ are unchanged.

Example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + u(t)$$

Changing coordinates to

$$z(t) = Tx(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} x(t)$$

we can represent the same map from u to y by

$$\dot{z}(t) = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} -1 & 2 \end{bmatrix} z(t) + u(t)$$

System equivalence

When are two systems are equivalent?

Theorem: Suppose (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are realizations for G_1 and G_2 respectively. Then

$$G_1 \text{ and } G_2 \text{ are equivalent} \quad \Longleftrightarrow \quad \begin{array}{l} C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2 \text{ for all } t \\ \text{and } D_1 = D_2 \end{array}$$

Proof

We have, for any realization (A, B, C, D)

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

The $\Longleftarrow$ direction follows immediately.

For the $\Longrightarrow$ direction, clearly $D_1 = D_2$, since $D_1 u(0) = D_2 u(0)$ for all $u(0)$.

We need to show that

$$\begin{array}{c} \displaystyle\int_0^t \big(C_1 e^{A_1(t-\tau)} B_1 - C_2 e^{A_2(t-\tau)} B_2\big) u(\tau)\, d\tau = 0 \\ \text{for all functions } u \text{ and for all } t \end{array} \quad \Longrightarrow \quad \begin{array}{c} C_1 e^{A_1 t} B_1 - C_2 e^{A_2 t} B_2 = 0 \\ \text{for all } t \end{array}$$

System equivalence 2

Proof continued

We want to show

$$\int_0^t F(t-\tau)u(\tau)\,d\tau = 0 \text{ for all } u, t \qquad \Longrightarrow \qquad F(t) = 0 \text{ for all } t$$

Compare this with

$$Ax = 0 \text{ for all } x \qquad \Longrightarrow \qquad A = 0$$

We will prove the case when F is scalar valued.

To show a contradiction, assume the above integral is zero for all u and t, yet there is some $t_0 \geq 0$ for which $F(t_0) \neq 0$. Pick

$$u(t) = F(t_0 + 1 - t)$$

and choose $t = t_0 + 1$. This gives $u(1) \neq 0$, and

$$\int_0^{t_0+1} F(t_0 + 1 - \tau)u(\tau)\,d\tau = \int_0^{t_0+1} |u(\tau)|^2\,d\tau > 0$$

which contradicts our assumption that the above integral is zero.

The proof in the matrix valued case is similar.

Removing uncontrollable states

The *dynamic order* or *state-dimension* of a state-space system is the dimension n of the generator matrix A.

If a system is not controllable, then there exists an equivalent lower-order realization.

Theorem: If $\dim(\mathcal{C}_{AB}) = r$, then we can choose coordinates so that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CT^{-1} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \qquad \bar{D} = D$$

where $\bar{A}_{11} \in \mathbb{R}^{r \times r}$, $\bar{B}_1 \in \mathbb{R}^{r \times m}$.

The lower-order system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)$ is equivalent to (A, B, C, D), and is controllable.

Notes

- This representation is called *controllability form*.
- Equivalence follows from the representation, because

$$\begin{aligned} \bar{C}e^{\bar{A}t}\bar{B} &= \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} e^{\bar{A}_{11}t} & ? \\ 0 & ? \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \\ &= \bar{C}_1 e^{\bar{A}_{11}t} \bar{B}_1 \end{aligned}$$

Removing uncontrollable states

Example

The 2nd order state-space system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-1 & -3\\ 0 & -2\end{bmatrix} x(t) + \begin{bmatrix}1\\0\end{bmatrix} u(t)\\ y(t) &= \begin{bmatrix}-1 & 0\end{bmatrix} x(t)\end{aligned}$$

represents the same map as the 1st order system

$$\begin{aligned}\dot{z}(t) &= -z(t) + u(t)\\ y(t) &= -x(t)\end{aligned}$$

The state component x_2 is uncontrollable. With initial condition $x(0) = 0$, the state component $x_2(t) = 0$ for all t.

Proof

We first show that the controllable subspace is A-invariant.

$$x \in \mathcal{C}_{AB} \qquad \Longrightarrow \qquad Ax \in \mathcal{C}_{AB}$$

This holds because, if $x \in \mathcal{C}_{AB}$, then

$$x \in \operatorname{image} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}.$$

Hence there exist vectors $w_1, w_2, \dots, w_n$, such that

$$x = Bw_1 + ABw_2 + \cdots + A^{n-1}Bw_n$$

and therefore

$$Ax = ABw_1 + A^2Bw_2 + \cdots + A^nBw_n.$$

But A^n is a linear combination of $I, A, A^2, \dots, A^{n-1}$

$$A^n = \mu_0 + \mu_1 A + \mu_2 A^2 + \cdots + \mu_{n-1}A^{n-1}$$

by the Cayley-Hamilton theorem. Hence Ax is the linear combination

$$Ax = B(\mu_0 w_n) + AB(\mu_1 w_n + w_1) + \cdots + A^{n-1}B(\mu_{n-1}w_n + w_{n-1})$$

and thus $Ax \in \mathcal{C}_{AB}$ also.

Proof continued

Now choose coordinates $z = Tx$ such that

$$\mathcal{C}_{AB} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^n \; ; \; z_2 = 0 \right\}$$

Note that $\dim(\mathcal{C}_{\mathcal{AB}}) = r$, and $z_1 \in \mathbb{R}^r$.

Partition TAT^{-1} compatibly with (z_1, z_2). Then

$$TAT^{-1}z = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathcal{C}_{AB} \qquad \text{for all } z \in \mathcal{C}_{AB}$$

This holds if and only if

$$\begin{aligned} & \bar{A}_{21}z_1 + \bar{A}_{22}z_2 = 0 && \text{for all } z \in \mathcal{C}_{AB} \\ \iff \quad & \bar{A}_{21}z_1 = 0 && \text{for all } z_1 \in \mathbb{R}^r \\ \iff \quad & \bar{A}_{21} = 0 \end{aligned}$$

Removing unobservable states

If $\dim(\mathcal{N}_{AB}) = n - r$, then we can choose coordinates so that

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$
$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{r \times r}$, $C_1 \in \mathbb{R}^{p \times r}$.

The lower-order system (A_{11}, B_1, C_1, D) is equivalent to (A, B, C, D), and is observable.

This representation is called *observability form*.

Proof

As for controllability, noting that the unobservable subspace is A-invariant.

Duality

The ideas of controllability and observability are called *dual*.

$$(C, A) \text{ is observable} \qquad \Longleftrightarrow \qquad (A^*, C^*) \text{ is controllable}$$

Another characterization of equivalence

Theorem: Suppose (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are realizations for G_1 and G_2 respectively. Then

$$G_1 \text{ and } G_2 \text{ are equivalent} \quad \iff \quad \begin{array}{l} C_1A_1^kB_1 = C_2A_2^kB_2 \quad \text{for all } k \geq 0 \\ \text{and } D_1 = D_2 \end{array}$$

The matrices $CB, CAB, CA^2B, \ldots$ are called the *Markov parameters* for G.

Proof: The $\Longleftarrow$ direction follows immediately from the previous lemma, since

$$Ce^{At}B = CB + CABt + CA^2B\frac{t^2}{2} + \cdots$$

For the $\Longrightarrow$ direction, we know

$$\begin{aligned} & C_1e^{A_1t}B_1 = C_2e^{A_2t}B_2 && \text{for all } t \\ \Longrightarrow \quad & \frac{d^k}{dt^k}C_1e^{A_1t}B_1 = \frac{d^k}{dt^k}C_2e^{A_2t}B_2 && \text{for all } t \text{ and } k \\ \Longrightarrow \quad & C_1A_1^ke^{A_1t}B_1 = C_2A_2^ke^{A_2t}B_2 && \text{for all } t \text{ and } k \\ \Longrightarrow \quad & C_1A_1^kB_1 = C_2A_2^kB_2 && \text{for all } k \end{aligned}$$

with the last equality following from the previous one at $t = 0$.

Minimal realizations

A realization (A, B, C, D) for a system G is called *minimal* if there does not exist a realization for G with smaller state dimension.

Theorem:

(A, B, C, D) is minimal $\iff$ (C, A) is observable and (A, B) is controllable

Notes

- We have already shown the $\implies$ direction.
- We will use the equality of the Markov parameters to prove the $\impliedby$ direction.
- The minimum n for which a realization exists is a property of the map G.

Proof

We need to show the $\Longleftarrow$ direction. Suppose (A, B, C, D) is controllable and observable, and $A \in \mathbb{R}^{n \times n}$. We will show that if (A_1, B_1, C_1, D_1) is an equivalent realization, then it must have order at least n.

We know $CA^kB = C_1A_1^kB_1$ for all $k \geq 0$. Hence

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{n-1} \end{bmatrix} \begin{bmatrix} B_1 & A_1B_1 & \cdots & A_1^{n-1}B_1 \end{bmatrix}$$

which is $\qquad O_{CA}C_{AB} = O_{C_1A_1}C_{A_1B_1}$

For any two matrices P and Q, we have *Sylvester's inequality*:

$$\operatorname{rank}(P) + \operatorname{rank}(Q) - n \leq \operatorname{rank}(PQ) \leq \min\bigl\{\operatorname{rank}(P), \operatorname{rank}(Q)\bigr\}$$

We know that $\operatorname{rank}(O_{CA}C_{AB}) \geq n$, from the left Sylvester inequality.

This implies that $\operatorname{rank}(O_{C_1A_1}C_{A_1B_1}) \geq n$, which implies that $\operatorname{rank}(O_{C_1A_1}) \geq n$ and $\operatorname{rank}(C_{A_1B_1}) \geq n$ from the right Sylvester inequality. Hence $O_{C_1A_1}$ has at least n columns and $C_{A_1B_1}$ has at least n rows, and therefore A_1 is at least $n \times n$.

Transfer functions

Recall the Laplace transform of f

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

- The Laplace transform is a linear map.
- if $\dot{f}(t)$ has a Laplace transform, then it is given by $s\hat{f}(s) - f(0)$.

Applying the Laplace transform to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad \text{with initial condition } x(0) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

gives

$$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s) \end{aligned}$$

and

$$\hat{y}(s) = \hat{G}(s)u(s) \qquad \text{where } \hat{G}(s) = C(sI - A)^{-1}B + D$$

Write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) := C(sI - A)^{-1}B + D.$$

Transfer functions

The function $\hat{G} : \mathbb{C} \to \mathbb{C}^{p \times m}$ is called the *transfer function*:

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

Rational functions

- A scalar function $\hat{g} : \mathbb{C} \to \mathbb{C}$ is called *rational* if

$$\hat{g}(s) = \frac{b_m s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

  It is called *real-rational* if the coefficients are real.

- $\hat{g}$ is called *proper* if $n \geq m$, and *strictly proper* if $n > m$.

Rational functions

- We call the matrix-valued function $\hat{G}$ *rational* if each of its entries is rational.
- The function $\hat{G}$ corresponding to a state-space systems is rational, since

$$\left[(sI - A)^{-1}\right]_{ij} = \frac{1}{\det(sI - A)} \times \text{cofactor of element } ij$$

where each cofactor is the determinant of a submatrix of $sI - A$.

- We call $\hat{G}$ proper if each of its entries is proper.

Equivalence of transfer functions

Given G_1 and G_2 defined by state-space representations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) respectively,

$$G_1 \text{ and } G_2 \text{ are equivalent} \quad \Longleftrightarrow \quad \hat{G}_1(s) = \hat{G}_2(s) \text{ for all } s$$

Proof

We know

$$G_1 \text{ and } G_2 \text{ are equivalent} \quad \Longleftrightarrow \quad \begin{array}{l} C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2 \text{ for all } t \\ \text{and } D_1 = D_2 \end{array}$$

Since the Laplace transform of e^{At} is $(sI - A)^{-1}$, this is equivalent to

$$C_1(sI - A_1)^{-1}B_1 = C_2(sI - A_2)^{-1}B_2 \quad \text{for all } s \qquad \text{and} \qquad D_1 = D_2$$

which holds if and only if

$$C_1(sI - A_1)^{-1}B_1 + D_1 = C_2(sI - A_2)^{-1}B_2 + D_2$$

(The 'if' part follows by equality as $s \to \infty$.)

Realizations for scalar systems

Given a scalar-valued (often called SISO) strictly proper transfer function $\hat{g}$

$$\hat{g}(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

there exists a state-space realization (A, B, C, D) which has order n.

Proof

It is

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_0 & \cdots & c_{n-1} \end{bmatrix} \qquad D = 0$$

Non-strictly proper $\hat{g}$

If $\hat{g}$ is proper but not strictly proper, we can write it as

$$\hat{g}(s) = \hat{g}_1(s) + D$$

where $\hat{g}_1$ is strictly proper.

Realizations

To realize a matrix-valued transfer function $\hat{G}$, we can do so in blocks.

Columns

Suppose

$$\hat{G}(s) = \begin{bmatrix} \hat{G}_1(s) & \hat{G}_2(s) \end{bmatrix}$$

and we have realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) for $\hat{G}_1$ and $\hat{G}_2$.

Then a realization for G is

$$\begin{bmatrix} \hat{G}_1(s) & \hat{G}_2(s) \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right]$$

Rows

Suppose $\hat{G}(s) = \begin{bmatrix} \hat{G}_1(s) \\ \hat{G}_2(s) \end{bmatrix}$. Then a realization for G is

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right]$$

Realizations 2

A procedure for realization of a rational transfer matrix $\hat{G}$ is

1. Realize each element $\hat{G}_{ij}$, which is a scalar transfer function.
2. Realize the columns.
3. Realize the row of columns.

Caveat

The resulting realization may be non-minimal. For example,

$$\hat{G}(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \end{bmatrix}$$

The previous construction leads to

$$\hat{G}(s) = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 1 & 2 & 0 & 0 \end{array}\right]$$

but a lower-order realization is

$$\hat{G}(s) = \left[\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 1 & 0 & 0 \end{array}\right]$$

Representation of systems

- View systems as linear operators on signal spaces. The map between inputs and outputs defines the system.
- Every proper rational transfer matrix has a state-space realization.
- Every state-space system has a proper transfer function representation.

Platonic theory of systems

- Analogous to the idea of *rank* of a matrix, we have the notion of *order* of a linear system.
- It can go wrong in similar ways; e.g.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & -3 \\ 0.1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -1 & 0 \end{bmatrix} x(t) \end{aligned}$$

$$C_{AB} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0.1 \end{bmatrix} \qquad \text{which has singular values } \sigma = \begin{bmatrix} 1.41 & 0 \\ 0 & 0.07 \end{bmatrix}$$

- We need a notion of approximation for systems. More later...