Engr210a Lecture 12: LFTs and stability

- 2-input 2-output framework
- Example problem formulations
- Linear fractional transformations
- Well-posedness
- Realizability
- Internal stability
- Input-output characterization of internal stability
2-input 2-output framework

Inputs

- Actuator inputs $u$ are those inputs to the system that can be manipulated by the controller.
- Exogenous inputs $w$ are all other inputs.

Outputs

- Regulated outputs $z$ are every output signal from the model.
- Sensed outputs are those outputs which are accessible to the controller.

Notes

- Objective is to write all specifications in terms of $z$ and $w$. 

**Command inputs and diagnostic outputs**

Formulate the above as
Example: the regulator

Formulate the above as

\[
\begin{align*}
z_1 &= y_p \\
z_2 &= u
\end{align*}
\]

The plant \( P \) is given by

\[
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
P_0 & 0 & P_0 \\
0 & 0 & 1 \\
P_0 & 1 & P_0
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2 \\
u
\end{bmatrix}
\]

Suppose \( P_0 \) is

\[
\dot{x} = Ax + Bq \\
r = Cx + Dq
\]

Substituting

\[
\begin{align*}
z_2 &= u \\
q &= w_1 + w_2 \\
z_1 &= r \\
y &= r + w_2
\end{align*}
\]

leads to

\[
P = \begin{bmatrix}
A & B & 0 & B \\
C & D & 0 & D \\
0 & 0 & 0 & I \\
C & D & I & D
\end{bmatrix}
\]
Example: a tracking problem

\[ r \rightarrow K \rightarrow P_0 \rightarrow e \]

\[ r = w_1 \]

\[ z_1 = e \]

\[ z_2 = u \]

\[ y \rightarrow K \rightarrow u \]

\[ u = P_0 \]

\[ n_{\text{proc}} = w_2 \]

\[ n_{\text{sensor}} = w_3 \]

\[ + \]

\[ - \]
Linear fractional transformations

Suppose \( P \) and \( K \) are state-space systems with

\[
\begin{bmatrix}
  z \\
  y
\end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\begin{bmatrix}
  w \\
  u
\end{bmatrix}
\]

where
\[
\hat{P} = \begin{bmatrix}
  A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22}
\end{bmatrix}
\]

and

\[
u = Ky\]

where
\[
\hat{K} = \begin{bmatrix}
  A_K & B_K \\ C_K & D_K
\end{bmatrix}
\]

The following interconnection is called the (lower) \textit{star-product} of \( P \) and \( K \), or the (lower) \textit{linear-fractional transformation} (LFT).

The map from \( w \) to \( z \) is given by

\[
S(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
\]
Linear fractional transformations

The following interconnection is called the (lower) star-product of \( P \) and \( K \), or the (lower) linear-fractional transformation (LFT).

\[
\begin{align*}
\begin{array}{c}
\text{\( P \)} \\
\text{\( y \)} \\
\text{\( u \)} \\
\\hline
\text{\( K \)} \\
\end{array}
\end{align*}
\]

Equivalently
Linear fractional transformations

Lower LFT

\[
S(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
\]

Upper LFT

\[
\overline{S}(P, K) = P_{22} + P_{21}Q(I - P_{11}Q)^{-1}P_{12}
\]

Star Product

\[
S(P, K) = \begin{bmatrix}
S(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\
K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \overline{S}(K, P_{22})
\end{bmatrix}
\]
General problem

A very general and useful way to formulate control problems is the following.

\[
\begin{align*}
\text{minimize} & \quad \| S(P, K) \| \\
\text{subject to} & \quad \text{The closed-loop is stable}
\end{align*}
\]

Notes

- Many different norms can be used; the two most common are the $H_2$ and $H_\infty$ norm.
- Robustness specifications can also be put into this form; we will see much more later.
Well-posedness

The state-space equations are

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\
\dot{x}_K(t) &= A_Kx(t) + B_Ky(t) \\
z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \\
y(t) &= C_2x(t) + D_{21}w(t) + D_{22}u(t)
\end{align*}
\]

For linear operators or transfer functions, we have

\[
S(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
\]

**Caveat:** invertibility here is as a transfer matrix, not as a bounded operator.

In state-space this is

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_K(t)
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
0 & A_K
\end{bmatrix} \begin{bmatrix}
x(t) \\
x_K(t)
\end{bmatrix} + \begin{bmatrix}
B_2 & 0 \\
0 & B_K
\end{bmatrix} \begin{bmatrix}
u(t) \\
y(t)
\end{bmatrix} + \begin{bmatrix}
B_1 \\
0
\end{bmatrix} w(t)
\]

\[
z(t) = \begin{bmatrix}
C_1 & 0
\end{bmatrix} \begin{bmatrix}
x(t) \\
x_K(t)
\end{bmatrix} + \begin{bmatrix}
D_{12} & 0
\end{bmatrix} \begin{bmatrix}
u(t) \\
y(t)
\end{bmatrix} + D_{11}w(t)
\]

where

\[
\begin{bmatrix}
I & -D_K \\
-D_{22} & I
\end{bmatrix} \begin{bmatrix}
u(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
0 & C_K \\
C_2 & 0
\end{bmatrix} \begin{bmatrix}
x(t) \\
x_K(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
D_{21}
\end{bmatrix} w(t)
\]

We need to solve these equations for \(u(t)\) and \(y(t)\).
Well-posedness

The LFT interconnection is called *well-posed* if unique solutions exist for $u$ and $y$, given

- any initial conditions $x(0)$ and $x_K(0)$,
- any smooth input function $w$,
- any small perturbations to the state-space matrices for $P$ and $K$.

**Theorem**

The LFT interconnection is well-posed $\iff I - D_{22}D_K$ is invertible.

**Notes**

- This follows from
  \[
  \begin{bmatrix}
    I & -D_K \\
    -D_{22} & I
  \end{bmatrix}
  \begin{bmatrix}
    u(t) \\
    y(t)
  \end{bmatrix}
  =
  \begin{bmatrix}
    0 & C_K \\
    C_2 & 0
  \end{bmatrix}
  \begin{bmatrix}
    x(t) \\
    x_K(t)
  \end{bmatrix}
  +
  \begin{bmatrix}
    0 \\
    D_{21}
  \end{bmatrix}
  w(t)
  \]

- Note that
  \[
  \begin{bmatrix}
    I & -D_K \\
    -D_{22} & I
  \end{bmatrix}^{-1}
  =
  \begin{bmatrix}
    I + D_K Q D_{22} & D_K Q \\
    Q D_{22} & Q
  \end{bmatrix}
  \]
  where $Q = (I - D_{22}D_K)^{-1}$.

- In frequency domain: $\lim_{w \to \infty} \hat{P}_{22}(j\omega) = D_{22}$ and $\lim_{w \to \infty} \hat{K}(j\omega) = D_K$.
Well-posedness

- The interconnection is well-posed iff $I - D_{22}D_K$ is invertible.
- The interconnection is well-posed iff $I - D_KD_{22}$ is invertible.
- In frequency domain, we have $\lim_{w \to \infty} \hat{P}_{22}(j\omega) = D_{22}$ and $\lim_{w \to \infty} \hat{K}(j\omega) = D_K$.
- If $P$ and $K$ are rational, then the LFT interconnection is well-posed if and only if there exists $w \in \mathbb{R}$ such that
  $$\det(I - \hat{P}_{22}(j\omega)\hat{K}(j\omega)) \neq 0$$
- If $D_K = 0$, that is if $\hat{K}$ is strictly proper, then the system is well-posed.
- If $D_{22} = 0$, that is if $\hat{P}_{22}$ is strictly proper, then the system is well-posed.

Notes

- We require well-posedness so that the system equations make sense.
- Physical systems are always well-posed; roughly, if $\hat{P}$ is a physical system then $\hat{P}$ must be strictly proper.
- Well-posedness says nothing about stability.
Realizability

A general problem can be written as

\[ \text{minimize} \quad \| H \| \]
\[ \text{subject to} \quad H = S(P, K) \text{ for some } \hat{K} \in RP \]

The closed-loop is stable

Notes

- \( H = S(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \). The map from \( K \) to \( H \) is nonlinear, so we have a nonlinear function of \( K \) to minimize.

- Instead, focus on the set of possible \( H \).

Alternative formulation

Let \( \mathcal{H}_{rlzbl} = \{ \hat{H} \in RP : H = S(P, K) \text{ for some } \hat{K} \in RP \} \).

\[ \text{minimize} \quad \| H \| \]
\[ \text{subject to} \quad H \in \mathcal{H}_{rlzbl} \]

The closed-loop is stable
Theorem

Suppose $P_{22}$ is strictly proper. Then the set $\mathcal{H}_{rlzbl}$ is affine.

Proof

- We know $H = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$. Let $R = (I - P_{22}K)^{-1}$. This map is one-to-one, since $K = (I + RP_{22})^{-1}R$, and $I + RP_{22}$ is always invertible (in $RP$), since $P_{22}$ is strictly proper.

- So given $\hat{K} \in RP$ we can construct $\hat{R} \in RP$, and vice-versa. Hence

$$\mathcal{H}_{rlzbl} = \left\{ P_{11} + P_{12}RP_{21} : \hat{R} \in RP \right\}$$

- Suppose $H_a, H_b \in \mathcal{H}_{rlzbl}$. We need to show that for any $\lambda \in \mathbb{R}$,

$$\lambda H_a + (1 - \lambda) H_b \in \mathcal{H}_{rlzbl}$$

Let $R_a$ and $R_b$ be such that $H_a = P_{11} + P_{12}R_aP_{21}$ and $H_b = P_{11} + P_{12}R_bP_{21}$. Choose $R_\lambda = \lambda R_a + (1 - \lambda) R_b$. Then

$$\lambda H_a + (1 - \lambda) H_b = P_{11} + P_{12}R_\lambda P_{21}$$
Realizability

\[ z \rightarrow + \rightarrow P_{11} \rightarrow w \]

\[ y \rightarrow K \rightarrow u \rightarrow + \rightarrow P_{21} \]

\[ P_{12} \rightarrow y \rightarrow K \rightarrow u \rightarrow + \rightarrow P_{21} \]

\[ z \rightarrow + \rightarrow P_{11} \rightarrow w \]

\[ y \rightarrow K \rightarrow u \rightarrow + \rightarrow P_{21} \]

\[ P_{12} \rightarrow y \rightarrow K \rightarrow u \rightarrow + \rightarrow P_{21} \]
Realizability

The general optimization problem is

\[
\begin{align*}
\text{minimize} & \quad \|H\| \\
\text{subject to} & \quad H \in \mathcal{H}_{rlzbl} \\
& \quad \text{The closed-loop is stable}
\end{align*}
\]

The set \(\mathcal{H}_{rlzbl}\) is

\[
\mathcal{H}_{rlzbl} = \{ P_{11} + P_{12}RP_{21} ; \hat{R} \in RP \}
\]

Equivalent problem

\[
\begin{align*}
\text{minimize} & \quad \|P_{11} + P_{12}RP_{21}\| \\
\text{subject to} & \quad \text{The closed-loop is stable}
\end{align*}
\]

Notes

- \(\mathcal{H}_{rlzbl}\) is convex, since it is affine.
- Optimization subject to the constraint that \(H \in \mathcal{H}_{rlzbl}\) may be tractable.
- Once \(R\) has been found, construct \(K\) from \(K = (I + RP_{22})^{-1}R\).
**Internal stability**

The system interconnection is called *internally stable* if, for every initial condition \(x(0)\) and \(x_K(0)\),

\[
\lim_{t \to \infty} x(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} x_K(t) = 0
\]

when \(w = 0\).

• We know

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_K(t)
\end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x(t) \\
x_K(t) \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u(t) \\
y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t)
\]

\[
\begin{bmatrix}
I & -D_K \\
-D_{22} & I
\end{bmatrix} \begin{bmatrix} u(t) \\
y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\
x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)
\]

• The dynamics of the interconnected system are

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_K(t)
\end{bmatrix} = A_{cl} \begin{bmatrix} x(t) \\
x_K(t) \end{bmatrix}
\]

where

\[
A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}
\]

• Hence the system is internally stable if and only if \(I - D_{22}D_K\) is invertible and \(A_{cl}\) is Hurwitz.
Internal stability

Suppose $\hat{P} \in RP$; that is $\hat{P}$ is a real-rational proper transfer function. Then

$$P \text{ is stable} \iff \hat{P} \in H_\infty$$

Exponential stability of the state then follows if the state-space realization for $P$ is controllable and observable.

Linear Fractional Transformations

The map from $w$ to $z$ is given by

$$S(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Is $S(\hat{P}, \hat{K}) \in H_\infty$ equivalent to exponential stability of the states when the realizations of $P$ and $K$ are controllable and observable?

Answer: No. e.g pick $P_{12} = 0$. 
Input-output characterization of internal stability

Consider the feedback loop:

\[
\begin{bmatrix}
    v_1 \\
    v_2
\end{bmatrix} = W \begin{bmatrix}
    d_1 \\
    d_2
\end{bmatrix}
\quad \text{where} \quad W = \begin{bmatrix}
    (I - KG)^{-1} & (I - KG)^{-1}K \\
    G(I - KG)^{-1} & G(I - KG)^{-1}K
\end{bmatrix}
\]

Inject actuator and sensor noise \(d_1\) and \(d_2\). Then

- A state-space system is called *stabilizable* if for any initial condition \(x(0)\) in the uncontrollable subspace, the state decays to zero.

- Similarly, a state-space system is called *detectable* if for any initial condition \(x(0)\) in the unobservable subspace, the state decays to zero.

Suppose the realizations for \(P_{22}\) and \(K\) are stabilizable and detectable. Then the above feedback loop is internally stable if and only if \(\dot{W} \in RH_\infty\).
Input-output characterization of internal stability

Suppose the realizations for \( P_{22} \) and \( K \) are stabilizable and detectable. Then the above feedback loop is internally stable if and only if

\[
\begin{bmatrix}
(I - KG)^{-1} & (I - KG)^{-1}K \\
G(I - KG)^{-1} & G(I - KG)^{-1}K
\end{bmatrix} \in RH_\infty
\]

- For scalar \( P_{22} \) and \( K \), this is equivalent to the statement that there are no unstable pole-zero cancellations.

- The above definition is valid in the multivariable case also, when zeros are not clearly defined.

- Any sensible design problem would include signals \( d_1 \) and \( d_2 \) as part of \( w \) and signals \( v_1 \) and \( v_2 \) as part of \( z \).
Example: unstable pole-zero cancellations

Consider the plant-controller pair

\[ \hat{G}(s) = \frac{10 - s}{(s + 10)s^2} \quad \hat{K}(s) = \frac{-3(12 + 11s)}{10 - s} \]

which has an unstable pole-zero cancellation.

Consider the tracking problem

\[ \hat{y}(s) = \frac{-3(12 + 11s)}{(s + 4)(s + 3)^2} \hat{r}(s) \]

But

\[
W = \begin{bmatrix}
\frac{(10+s)s^2}{(s+4)(s+3)^2} & \frac{3(12+11s)(10+s)s^2}{(s-10)(s+4)(s+3)^2} \\
\frac{10-s}{(s+4)(s+3)^2} & \frac{-3(12+11s)}{(s+4)(s+3)^2}
\end{bmatrix}
\]

and the pole-zero cancellation shows as instability of \( W \).
Internal stability and LFTs

Suppose $P$ and $K$ are state-space systems with

$$
\begin{bmatrix}
  z \\
  y
\end{bmatrix}
= 
\begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
w \\
u
\end{bmatrix}
$$

where

$$
\hat{P} = \begin{bmatrix}
  A & B_1 & B_2 \\
  C_1 & D_{11} & D_{12} \\
  C_2 & D_{21} & D_{22}
\end{bmatrix}
$$

and

$$
\hat{P}_{22} = \begin{bmatrix}
  A_{22} & B_2 \\
  C_2 & D_{22}
\end{bmatrix}
$$

and

$$
u = Ky$$

where

$$
\hat{K} = \begin{bmatrix}
  A_K & B_K \\
  C_K & D_K
\end{bmatrix}
$$

Theorem

Suppose $(A, B_2)$ is stabilizable and $(A, C_2)$ is detectable. Then

is internally stable

is internally stable