• Internal stability

• Stabilizing controllers

• Achievable closed-loop maps

• Interpolation

• Parametrization of stabilizing controllers

• Division and coprimeness

• Euclid’s algorithm

• The Bezout equation

• Coprime factorization in $H_{\infty}$. 
Alternative characterization of internal stability

This interconnection is equivalent to

\[
\begin{bmatrix}
  I & -K \\
  -P_{22} & I
\end{bmatrix}
\begin{bmatrix}
  r_1 \\
  r_2
\end{bmatrix}
= \begin{bmatrix}
  f_1 \\
  f_2
\end{bmatrix}
\]

Let

\[
Z = \begin{bmatrix}
  I & -K \\
  -P_{22} & I
\end{bmatrix}^{-1}
= \begin{bmatrix}
  (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\
  (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1}
\end{bmatrix}
\]

Suppose the realizations for \( P_{22} \) and \( K \) are stabilizable and detectable. Then

\[ Z \in H_\infty \iff \text{the interconnection is internally stable} \]
Stabilizing controllers

Controller $K$ is called *stabilizing* if the interconnection of $P_{22}$ and $K$ is internally stable.

Characterizations

Assume the realizations for $P_{22}$ and $K$ are stabilizable and detectable. Then

- $K$ is stabilizing if and only if
  \[
  \begin{bmatrix}
  I & -K \\
  -P_{22} & I
  \end{bmatrix}^{-1} = \begin{bmatrix}
  (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\
  (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1}
  \end{bmatrix}
  \]
  is stable

- Special case: if $K$ is stable, then
  \[
  K \text{ is stabilizing} \iff (I - P_{22}K)^{-1}P_{22} \text{ is stable}
  \]

  Proof: Note that
  \[
  \begin{bmatrix}
  I & -K \\
  -P_{22} & I
  \end{bmatrix}^{-1} = \begin{bmatrix}
  I + K(I - P_{22}K)^{-1}P_{22} & K(I + (I - P_{22}K)^{-1}P_{22}) \\
  (I - P_{22}K)^{-1}P_{22} & I + (I - P_{22}K)^{-1}P_{22}K
  \end{bmatrix}
  \]

- Another special case: if $P$ is stable, then
  \[
  K \text{ is stabilizing} \iff K(I - P_{22}K)^{-1} \text{ is stable}
  \]
Stable interconnections

Recall the set of realizable maps $H : w \rightarrow z$ is

$$\mathcal{H}_{rlzbl} = \{ \hat{H} \in RP ; \ H = S(P,K) \ \text{for some } \hat{K} \in RP \}$$

$$= \{ P_{11} + P_{12} \hat{R} P_{21} ; \hat{R} \in RP \}$$

Define the set

$$\mathcal{H}_{\text{stable}} = \left\{ \begin{array}{c} \hat{H} \in RP ; \ H = S(P,K) \ \text{for some } \hat{K} \in RP \\ \text{the interconnection is internally stable} \end{array} \right\}$$

the set of closed-loop maps achievable by stabilizing controllers.

**Theorem**

Suppose $P_{22}$ is proper. Then $\mathcal{H}_{\text{stable}}$ is affine.
**Theorem**

Suppose $P_{22}$ is proper. Then $\mathcal{H}_{\text{stable}}$ is affine.

**Proof**

- $H \in \mathcal{H}_{\text{stable}}$ if and only if $H = P_{11} + P_{12}RP_{21}$, and

  $$Z = \begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{bmatrix}$$

  is stable

  Substituting $R = K(I - P_{22}K)^{-1}$ gives

  $$Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix}$$

  Then $K = (I + RP_{22})^{-1}R$ is stabilizing if and only if $Z$ is stable.

- The map from $R$ to $Z$ is affine, and therefore the preimage of $H_{\infty}$ under this map is an affine set in $L_{\infty}$.

- Hence the set of $R$ such that $K = (I + RP_{22})^{-1}R$ is stabilizing is an affine set.

- The map from $R$ to $H$ is affine, and the image of an affine set under an affine map is affine.
Interpolation conditions

We have

\[ K = (I + RP_{22})^{-1}R \text{ is stabilizing} \iff Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix} \text{ is stable} \]

For scalar plant and controller \( \hat{P}_{22} \) and \( \hat{K}_{22} \), let \( T = RP_{22} \). Then

\[ K = (I + T)^{-1}TP_{22}^{-1} \text{ is stabilizing} \iff Z = \begin{bmatrix} I + T & TG^{-1} \\ (I + T)P_{22} & I + T \end{bmatrix} \text{ is stable} \]

Let \( z_1, \ldots, z_k \) be the unstable zeros and \( p_1, \ldots, p_m \) be the unstable poles of \( P_{22} \). Assume they are distinct. Then

\[ K = (I + T)^{-1}TP_{22}^{-1} \text{ is stabilizing} \iff \hat{T} \in H_\infty \]

\[ \hat{T}(p_i) = -1 \text{ for } i = 1, \ldots, m \]

\[ \hat{T}(z_i) = 0 \text{ for } i = 1, \ldots, k \]

relative degree of \( T \geq \) relative degree of \( P_{22} \).

Then the closed loop map is

\[ S(P, K) = P_{11} + \frac{P_{12}TP_{21}}{P_{22}} \]

Note that the maximum modulus principle then implies that \( \|Z_{11}\| \geq 1 \) and \( \|Z_{22}\| \geq 1 \) if \( P_{22} \) has RHP zeroes; hence weights are essential.
Optimization and interpolation

The general problem is

\[
\text{minimize} \quad \|H\| \\
\text{subject to} \quad H = \mathcal{S}(P, K) \text{ for some } \hat{K} \in RP
\]

The closed-loop is stable

Equivalent formulation for scalar \(P_{22}\)

Let \(z_1, \ldots, z_k\) be the unstable zeros and \(p_1, \ldots, p_m\) be the unstable poles of \(P_{22}\). Assume they are distinct.

\[
\text{minimize} \quad \|P_{11} + P_{12}TP_{22}^{-1}P_{21}\| \\
\text{subject to} \quad T \in H_{\infty}^{1 \times 1} \\
\hat{T}(p_i) = -1 \text{ for } i = 1, \ldots, m \]

\[
\hat{T}(z_i) = 0 \text{ for } i = 1, \ldots, k
\]

relative degree of \(T \geq \) relative degree of \(P_{22}\).

This is an example of a Nevanlinna-Pick interpolation problem. In general, these problems are hard to solve (but it can be done).
Stabilizing controllers for stable plants

Suppose $P$ is stable. Then

$$K \text{ is stabilizing } \iff K = \left( I + RP_{22} \right)^{-1} R \text{ for some stable } R$$

Then

$$\mathcal{H}_{\text{stable}} = \left\{ P_{11} + P_{12}RP_{21} ; \hat{R} \in H_{\infty} \right\}$$

Proof

$Z$ is stable if and only if $R$ is stable, since

$$Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix}$$

Notes

• If $P$ is stable, then the above gives a simple parametrization of all stabilizing controllers.

• What about when $P$ is unstable? We need the notion of coprime factorization.
Optimization for stable $P$

The general problem is

$$\text{minimize} \quad \|H\|$$

subject to \( H = S(P, K) \) for some \( \hat{K} \in RP \)

The closed-loop is stable

Equivalent formulation for stable $P$

$$\text{minimize} \quad \|P_{11} + P_{12}RP_{21}\|$$

subject to \( R \in H_\infty \)

Once the optimal $R$ is found, then the optimal $K$ is given by

$$K = (I + RP_{22})^{-1}R$$
Coprime Factorization

Suppose \( n, d \in \mathbb{Z} \) are integers. Then

\[
d \text{ divides } n \quad \text{ if there exists } q \in \mathbb{Z} \text{ such that } n = dq
\]

The integer \( d \) is called the greatest common divisor (gcd) of \( n, m \in \mathbb{Z} \) if

- \( d \) divides \( n \) and \( d \) divides \( m \).
- Every integer \( a \) that divides both \( n \) and \( m \) also divides \( d \).

\( n \) and \( m \) are called coprime if their gcd is 1.

Examples

- 10 and 21 are coprime.
- 12 and 21 are not coprime. Their gcd is 3.
Division

Given $n, m \in \mathbb{Z}$, and $n \leq m$. Then there exists a unique $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $r < n$ such that

$$m = nq + r$$

$q$ is the quotient, $r$ is the remainder.

Euclid’s algorithm

Euclid’s algorithm gives a way to find the gcd of $n, m \in \mathbb{Z}$.

$$a_0 = m; \quad b_0 = n; \quad k = 1;$$

Repeat \{ 

Find $q$ and $r$ so that $a_k = qb_k + r$;

$a_k = b_{k-1}; \quad b_k = r$

$k = k + 1;$$

} until $r = 0$.

The gcd is then $a_{k-1}$. 
Polynomials

Let $\mathbb{R}[s]$ be the set of polynomials in the variable $s$.

Suppose $n, d \in \mathbb{R}[s]$ are polynomials. Then

$$d \text{ divides } n \quad \text{ if there exists } q \in \mathbb{R}[s] \text{ such that } n = dq$$

The polynomial $d$ is called a *greatest common divisor* (gcd) of $n, m \in \mathbb{R}[s]$ if

- $d$ divides $n$ and $d$ divides $m$.
- Every $a \in \mathbb{R}[s]$ that divides both $n$ and $m$ also divides $d$.

$n$ and $m$ are called *coprime* if their gcd is a scalar.

Examples

- $(x - 1)(x - 2)$ and $(x - 3)$ are coprime.
- $(x - 1)(x^2 + 2)$ and $(x - 1)$ are not coprime. A gcd is any scalar multiple of $(x - 1)$.
Polynomials

Given two polynomials $n(s)$ and $m(s)$, we can apply Euclid’s algorithm to find their gcd.

Euclid’s algorithm

Euclid’s algorithm gives a way to find the gcd of $n, m \in \mathbb{R}[s]$.

\[
a_0 = m; \quad b_0 = n; \quad k = 1;
\]

Repeat \{
\[
\text{Find } q \text{ and } r \text{ so that } a_k = qb_k + r;
\]
\[
a_k = b_{k-1}; \quad b_k = r
\]
\[
k = k + 1;
\]
\} \text{ until } r = 0.

A gcd is then $a_{k-1}$.

\[
\begin{array}{c|c}
    a_k & b_k \\
    \hline
    s^4 + 5s^3 + 6s^2 + 3s + 1 & s^4 + 3s^3 + s + 2s^2 + 1 \\
    s^4 + 3s^3 + s + 2s^2 + 1 & 2s^3 + 4s^2 + 2s \\
    2s^3 + 4s^2 + 2s & -s^2 + 1 \\
    -s^2 + 1 & 4s + 4 \\
    4s + 4 & 0
\end{array}
\]
Euclid’s algorithm (∼ 300 B.C.)

$a_{k-1}$ is the gcd of $n$ and $m$.

Proof

• We have $a_0 = m$, $a_1 = n$, and $a_k = 0$, where

$$a_{i-2} = q_i a_{i-1} + a_i \quad \text{for } i = 2, \ldots, k$$

• We know $a_{k-1}$ divides $a_{k-2}$, and the above equation implies that if $a_i$ divides $a_{i-1}$ then $a_i$ divides $a_{i-2}$. Hence by induction, $a_{k-1}$ divides $a_0$ and $a_1$; that is, $a_{k-1}$ divides $m$ and $n$.

• Also, $a_i = a_{i-2} - q_i a_{i-1}$ implies that

$$a_i = xa_{i-2} + ya_{i-1} \quad \text{for some } x, y \in \mathbb{Z}.$$ 

for $i = 2, \ldots, k-1$. That is, $a_i$ is a linear combination of $a_{i-2}$ and $a_{i-1}$ where the coefficients are integers. By induction again, we have

$$a_{k-1} = xa_0 + ya_1$$

$$a_{k-1} = xm + yn \quad \text{for some } x, y \in \mathbb{Z}.$$ 

hence any divisor of $m$ and $n$ is also a divisor of $a_{k-1}$. Hence $a_{k-1}$ is a gcd.
The Bezout equation

The integers $m, n \in \mathbb{Z}$ are coprime if and only if there exists $x, y \in \mathbb{Z}$ such that

$$xm + yn = 1$$

This equation is called the *Bezout equation*.

Proof

The proof follows immediately from the above proof for Euclid's algorithm.

Notes

- Euclid’s algorithm works for
  - The integers $\mathbb{Z}$.
  - Polynomials $\mathbb{R}[s]$.
  - Scalar, stable, proper rational functions in $RH_\infty$.
  - Matrix-valued stable, proper rational functions in $RH_\infty$.
- The general algebraic structure for which this works is called a *ring*.
- The *if* direction is easy; e.g. for polynomials, if $m$ and $n$ have a common zero, then their cannot exist a solution to the Bezout equation.
Scalar stable proper transfer functions

Suppose \( m, n \in RH_{\infty}^{1 \times 1} \). Then

\[
d \text{ divides } n \quad \text{ if there exists } q \in RH_{\infty}^{1 \times 1} \text{ such that } n = dq
\]

Notes

- \( d \) divides \( n \) if and only if \( \frac{n}{d} \in RH_{\infty}^{1 \times 1} \)

Examples

- \( f_1(s) = \frac{s + 1}{(s + 2)^2} \) \quad \( f_2(s) = \frac{s - 1}{s + 1} \) \quad \( f_3(s) = \frac{s - 1}{(s + 1)^2} \)

- \( g_1(s) = \frac{s - 1}{s + 4} \) \quad \( g_2(s) = \frac{1}{3} \) \quad \( g_3(s) = \frac{s - 1}{(s + 2)^2} \)

- \( f_1 \) divides \( g_2 \) and \( g_3 \), but not \( g_1 \).
- \( f_2 \) divides \( g_1 \) and \( g_3 \), but not \( g_2 \).
- \( f_3 \) divides \( g_3 \), but not \( g_1 \) or \( g_2 \).
Scalar stable proper transfer functions

d \in RH_{\infty}^{1 \times 1} is called a greatest common divisor (gcd) of n, m \in RH_{\infty}^{1 \times 1} if

- d divides n and d divides m.
- Every a \in RH_{\infty}^{1 \times 1} that divides both n and m also divides d.

n and m are called coprime if d and d^{-1} are stable and proper for all gcids d.

Notes

- n and m are coprime if and only if they have no common zeros in the right-half-plane, or at infinity.

Examples

- n = \frac{s}{(s + 1)^2} and m = \frac{s - 1}{s + 1} are coprime. \(xm + yn = 1\) is satisfied for

  \[x = \frac{(2s + 4)(s + 1)^2}{s^3 + 3/2s^2 + 3s + 1/2}\quad\text{and}\quad y = \frac{(s - 1/2)(s + 1)^2}{s^3 + 3/2s^2 + 3s + 1/2}\]

- \(\frac{s - 1}{(s + 3)^2}\) and \(\frac{s - 2}{(s + 3)^2}\) are not coprime.
Coprime factorization

Rational numbers

Given $p \in \mathbb{Q}$, find $n, m \in \mathbb{Z}$ such that

$$p = \frac{n}{m} \quad \text{and } n, m \text{ are coprime}$$

Rational functions; factorization over $\mathbb{R}[s]$

Given $p \in RP^{1 \times 1}$ find $n, m \in \mathbb{R}[s]$ such that

$$p = \frac{n}{m} \quad \text{and } n, m \text{ are coprime}$$

$n, m$ always exist; just cancel any common zeros.

Rational functions; factorization over $RH_{\infty}^{1 \times 1}$

Given $p \in RP^{1 \times 1}$ find $n, m \in RH_{\infty}^{1 \times 1}$ such that

$$p = \frac{n}{m} \quad \text{and } n, m \text{ are coprime}$$

In contrast to above: $n, m$ must be stable proper transfer functions.
Coprime factorization over $RH_{\infty}^{1\times 1}$

Given $p \in RP_{1\times 1}^1$ find $n, m \in RH_{\infty}^{1\times 1}$ such that

$$p = \frac{n}{m} \quad \text{and } n, m \text{ are coprime}$$

Notes

- $n, m$ must be stable proper transfer functions.
- A coprime factorization always exists; make all stable poles of $p$ poles of $n$, all stable zeros of $p$ poles of $m$, and add zeros to $n$ and $m$ as necessary.

Example

Suppose $\hat{p}$ is

$$\hat{p}(s) = \frac{(s - 1)(s + 2)}{(s - 3)(s + 4)}$$

A coprime-factorization is

$$\hat{n}(s) = \frac{s - 1}{s + 4} \quad \hat{m}(s) = \frac{s - 3}{s + 2}$$
Coprime transfer functions in $RH_\infty$.

Suppose $M, N \in RH_\infty$, and let $D \in RH_\infty$ be square. Then

$D$ right-divides $N$ if there exists $Q \in RH_\infty$ such that $N = QD$

The square $D \in RH_\infty$ us called a right greatest common divisor of $M, N$ if

- $D$ right-divides $N$ and $D$ right-divides $M$.
- Every $A \in RH_{1 \times 1}$ that right-divides both $N$ and $M$ also right-divides $D$.

$N$ and $M$ are called right-coprime if $D$ and $D^{-1}$ are stable and proper for all gcds $D$.

The Bezout equation

$M, N \in RH_\infty$ are right-coprime if and only if there exists $X, Y \in RH_\infty$ such that

$$XM + YN = I$$
Coprime factorization in $RH_\infty$.

**Right-coprime factorization**

Given $P \in RP$, a factorization such that

- $P = NM^{-1}$
- $N, M \in RH_\infty$
- $N$ and $M$ are right-coprime

is called a right-coprime factorization of $P$.

**Left-coprime factorization**

Given $P \in RH_\infty$, a factorization such that

- $P = \tilde{M}^{-1}\tilde{N}$
- $\tilde{N}, \tilde{M} \in RH_\infty$
- $\tilde{N}$ and $\tilde{M}$ are left-coprime

is called a left-coprime factorization of $P$.

**Notes**

- Left and right coprime factorizations always exist.

**Example**

Suppose $P$ is

$$\hat{P}(s) = \frac{s}{(s + 1)(s - 1)}$$

A coprime-factorization is

$$N(s) = \frac{s}{(s + 1)^2} \quad M(s) = \frac{s - 1}{s + 1}$$
Stabilization via coprime factorization

Scalar example

Suppose $\hat{p}_{22} \in RH_{\infty}^{1 \times 1}$. Let

$$\hat{p}_{22}(s) = \frac{\hat{n}(s)}{\hat{m}(s)}$$

be a coprime factorization, and $\hat{x}, \hat{y} \in RH_{\infty}^{1 \times 1}$ satisfy the Bezout equation

$$\hat{x}(s)\hat{m}(s) - \hat{y}(s)\hat{n}(s) = 1$$

**Theorem**

$\hat{k}(s) = \frac{\hat{y}(s)}{\hat{x}(s)}$ is a stabilizing controller.

**Proof**

$$\hat{Z} = \left[ \begin{array}{cc} I & -\hat{k} \\ -\hat{p}_{22} & I \end{array} \right]^{-1} = \frac{1}{1 - \hat{k}\hat{p}_{22}} \left[ \begin{array}{cc} 1 & \hat{k} \\ \hat{p}_{22} & 1 \end{array} \right]$$

$$= \frac{1}{\hat{x}\hat{m} - \hat{y}\hat{n}} \left[ \begin{array}{cc} \hat{x}\hat{m} & \hat{y}\hat{m} \\ \hat{x}\hat{n} & \hat{x}\hat{m} \end{array} \right] = \left[ \begin{array}{cc} \hat{x}\hat{m} & \hat{y}\hat{m} \\ \hat{x}\hat{n} & \hat{x}\hat{m} \end{array} \right]$$

which is stable.
Every stabilizing controller

Suppose \( \hat{p}_{22} \in RH_{\infty}^{1\times1} \). Let \( \hat{p}_{22}(s) = \hat{n}(s)\hat{m}^{-1}(s) \) be a coprime factorization, and \( \hat{x}, \hat{y} \in RH_{\infty}^{1\times1} \) satisfy the Bezout equation \( \hat{x}(s)\hat{m}(s) - \hat{y}(s)\hat{n}(s) = 1 \).

**Theorem**

Every stabilizing controller has the form

\[
\hat{k} = \frac{\hat{y} - \hat{m}\hat{q}}{\hat{x} - \hat{n}\hat{q}}
\]

for some \( q \in RH_{\infty}^{1\times1} \).

**Proof**

The proof that \( \hat{k} \) is stabilizing is the same as before, since

\[
(\hat{x} - \hat{n}\hat{q})\hat{m} - (\hat{y} - \hat{m}\hat{q}) = 1
\]

Then

\[
\hat{Z} = \begin{bmatrix}
(\hat{x} - \hat{n}\hat{q})\hat{m} & (\hat{y} - \hat{m}\hat{q})\hat{m} \\
(\hat{x} - \hat{n}\hat{q})\hat{n} & (\hat{x} - \hat{n}\hat{q})\hat{m}
\end{bmatrix}
\]

which is stable.

We will prove that every \( \hat{k} \) has this form in the matrix case.