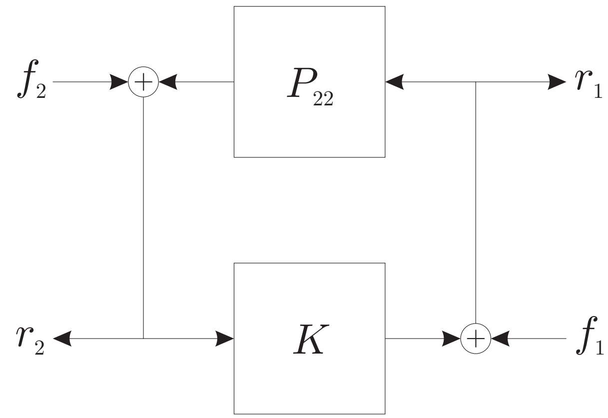
Engr210a Lecture 13: Internal stability and coprime factorization

- Internal stability
- Stabilizing controllers
- Achievable closed-loop maps
- Interpolation
- Parametrization of stabilizing controllers
- Division and coprimeness
- Euclid's algorithm
- The Bezout equation
- Coprime factorization in H_∞.

Alternative characterization of internal stability



This interconnection is equivalent to

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Let

$$Z = \begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{bmatrix}$$

Suppose the realizations for P_{22} and K are stabilizable and detectable. Then

$$Z \in H_\infty \quad \Longleftrightarrow \quad \text{the interconnection is internally stable}$$

Stabilizing controllers

Controller K is called *stabilizing* if the interconnection of P_{22} and K is internally stable.

Characterizations

Assume the realizations for P_{22} and K are stabilizable and detectable. Then

- K is stabilizing if and only if

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{bmatrix} \quad \text{is stable}$$

- Special case: if K is stable, then

$$K \text{ is stabilizing} \quad \Longleftrightarrow \quad (I - P_{22}K)^{-1}P_{22} \text{ is stable}$$

  Proof: Note that

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + K(I - P_{22}K)^{-1}P_{22} & K\big(I + (I - P_{22}K)^{-1}P_{22}\big) \\ (I - P_{22}K)^{-1}P_{22} & I + (I - P_{22}K)^{-1}P_{22}K \end{bmatrix}$$

- Another special case: if P is stable, then

$$K \text{ is stabilizing} \quad \Longleftrightarrow \quad K(I - P_{22}K)^{-1} \text{ is stable}$$

Stable interconnections

Recall the set of realizable maps $H : w \to z$ is

$$\begin{aligned}\mathcal{H}_{\mathsf{rlzbl}} &= \{\hat{H} \in RP \ ; \ H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP\} \\ &= \{P_{11} + P_{12}RP_{21} \ ; \ \hat{R} \in RP\}\end{aligned}$$

Define the set

$$\mathcal{H}_{\mathsf{stable}} = \left\{ \begin{aligned} \hat{H} \in RP \ ; \ H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP \\ \text{the interconnection is internally stable} \end{aligned} \right\}$$

the set of closed-loop maps achievable by stabilizing controllers.

Theorem

Suppose P_{22} is proper. Then $\mathcal{H}_{\mathsf{stable}}$ is affine.

Theorem

Suppose P_{22} is proper. Then $\mathcal{H}_{\text{stable}}$ is affine.

Proof

- $H \in \mathcal{H}_{\text{stable}}$ if and only if $H = P_{11} + P_{12}RP_{21}$, and

$$Z = \begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{bmatrix} \qquad \text{is stable}$$

  Substituting $R = K(I - P_{22}K)^{-1}$ gives

$$Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix}$$

  Then $K = (I + RP_{22})^{-1}R$ is stabilizing if and only if Z is stable.

- The map from R to Z is affine, and therefore the preimage of H_∞ under this map is an affine set in L_∞.

- Hence the set of R such that $K = (I + RP_{22})^{-1}R$ is stabilizing is an affine set.

- The map from R to H is affine, and the image of an affine set under an affine map is affine.

Interpolation conditions

We have

$$K = (I + RP_{22})^{-1}R \text{ is stabilizing} \quad \Longleftrightarrow \quad Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix} \text{ is stable}$$

For scalar plant and controller $\hat{P}_{22}$ and $\hat{K}_{22}$, let $T = RP_{22}$. Then

$$K = (I + T)^{-1}TP_{22}^{-1} \text{ is stabilizing} \quad \Longleftrightarrow \quad Z = \begin{bmatrix} I + T & TG^{-1} \\ (I + T)P_{22} & I + T \end{bmatrix} \text{ is stable}$$

Let $z_1, \ldots, z_k$ be the unstable zeros and $p_1, \ldots, p_m$ be the unstable poles of P_{22}. Assume they are distinct. Then

$$K = (I{+}T)^{-1}TP_{22}^{-1} \text{ is stabilizing} \quad \Longleftrightarrow \quad \begin{aligned} &\hat{T} \in H_\infty \\ &\hat{T}(p_i) = -1 \text{ for } i = 1, \ldots, m \\ &\hat{T}(z_i) = 0 \text{ for } i = 1, \ldots, k \\ &\text{relative degree of } T \geq \text{relative degree of } P_{22}. \end{aligned}$$

Then the closed loop map is $\underline{S}(P, K) = P_{11} + \dfrac{P_{12}TP_{21}}{P_{22}}$

Note that the maximum modulus principle then implies that $\|Z_{11}\| \geq 1$ and $\|Z_{22}\| \geq 1$ if P_{22} has RHP zeroes; hence weights are essential.

Optimization and interpolation

The general problem is

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H = \underline{S}(P,K) \text{ for some } \hat{K} \in RP \\ & \text{The closed-loop is stable} \end{array}$$

Equivalent formulation for scalar P_{22}

Let $z_1, \ldots, z_k$ be the unstable zeros and $p_1, \ldots, p_m$ be the unstable poles of P_{22}. Assume they are distinct.

$$\begin{array}{rl} \text{minimize} & \|P_{11} + P_{12} T P_{22}^{-1} P_{21}\| \\ \text{subject to} & T \in H_\infty^{1\times 1} \\ & \hat{T}(p_i) = -1 \text{ for } i = 1, \ldots, m \\ & \hat{T}(z_i) = 0 \text{ for } i = 1, \ldots, k \\ & \text{relative degree of } T \geq \text{relative degree of } P_{22}. \end{array}$$

This is an example of a *Nevanlinna-Pick* interpolation problem. In general, these problems are hard to solve (but it can be done).

Stabilizing controllers for stable plants

Suppose P is stable. Then

$$K \text{ is stabilizing} \quad \iff \quad K = (I + RP_{22})^{-1}R \text{ for some stable } R$$

Then

$$\mathcal{H}_{\text{stable}} = \left\{ P_{11} + P_{12}RP_{21} \; ; \; \hat{R} \in H_\infty \right\}$$

Proof

Z is stable if and only if R is stable, since

$$Z = \begin{bmatrix} I + RP_{22} & R \\ (I + P_{22}R)P_{22} & I + P_{22}R \end{bmatrix}$$

Notes

- If P is stable, then the above gives a simple parametrization of all stabilizing controllers.
- What about when P is unstable? We need the notion of *coprime factorization*.

Optimization for stable P

The general problem is

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP \\ & \text{The closed-loop is stable} \end{array}$$

Equivalent formulation for stable P

$$\begin{array}{rl} \text{minimize} & \|P_{11} + P_{12} R P_{21}\| \\ \text{subject to} & R \in H_\infty \end{array}$$

Once the optimal R is found, then the optimal K is given by

$$K = (I + RP_{22})^{-1} R$$

Coprimeness

Suppose $n, d \in \mathbb{Z}$ are integers. Then

$$d \text{ divides } n \qquad \text{if there exists } q \in \mathbb{Z} \text{ such that } n = dq$$

The integer d is called the *greatest common divisor* (gcd) of $n, m \in \mathbb{Z}$ if

- d divides n and d divides m.
- Every integer a that divides both n and m also divides d.

n and m are called *coprime* if their gcd is 1.

Examples

- 10 and 21 are coprime.
- 12 and 21 are not coprime. Their gcd is 3.

Division

Given $n, m \in \mathbb{Z}$, and $n \leq m$. Then there exists a unique $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $r < n$ such that

$$m = nq + r$$

q is the quotient, r is the remainder.

Euclid's algorithm

Euclid's algorithm gives a way to find the gcd of $n, m \in \mathbb{Z}$.

$a_0 = m; \quad b_0 = n; \quad k = 1;$

Repeat {

Find q and r so that $a_k = qb_k + r$;

$a_k = b_{k-1}; \quad b_k = r$

$k = k + 1;$

} until $r = 0$.

a_k	b_k
57	12
12	9
9	3
3	0

The gcd is then a_{k-1}.

Polynomials

Let $\mathbb{R}[s]$ be the set of polynomials in the variable s.

Suppose $n, d \in \mathbb{R}[s]$ are polynomials. Then

d divides n if there exists $q \in \mathbb{R}[s]$ such that $n = dq$

The polynomial d is called a *greatest common divisor* (gcd) of $n, m \in \mathbb{R}[s]$ if

- d divides n and d divides m.
- Every $a \in \mathbb{R}[s]$ that divides both n and m also divides d.

n and m are called *coprime* if their gcd is a scalar.

Examples

- $(x-1)(x-2)$ and $(x-3)$ are coprime.
- $(x-1)(x^2+2)$ and $(x-1)$ are not coprime. A gcd is any scalar multiple of $(x-1)$.

Polynomials

Given two polynomials $n(s)$ and $m(s)$, we can apply Euclid's algorithm to find their gcd.

Euclid's algorithm

Euclid's algorithm gives a way to find the gcd of $n, m \in \mathbb{R}[s]$.

$a_0 = m; \quad b_0 = n; \quad k = 1;$

Repeat {

Find q and r so that $a_k = q b_k + r$;

$a_k = b_{k-1}; \quad b_k = r$

$k = k + 1$;

} until $r = 0$.

A gcd is then a_{k-1}.

a_k	b_k
$s^4 + 5\,s^3 + 6\,s^2 + 3\,s + 1$	$s^4 + 3\,s^3 + s + 2\,s^2 + 1$
$s^4 + 3\,s^3 + s + 2\,s^2 + 1$	$2\,s^3 + 4\,s^2 + 2\,s$
$2\,s^3 + 4\,s^2 + 2\,s$	$-s^2 + 1$
$-s^2 + 1$	$4\,s + 4$
$4\,s + 4$	0

Euclid's algorithm ($\sim$ 300 B.C.)

a_{k-1} is the gcd of n and m.

Proof

- We have $a_0 = m$, $a_1 = n$, and $a_k = 0$, where

$$a_{i-2} = q_i a_{i-1} + a_i \qquad \text{for } i = 2, \ldots, k$$

- We know a_{k-1} divides a_{k-2}, and the above equation implies that if a_i divides a_{i-1} then a_i divides a_{i-2}. Hence by induction, a_{k-1} divides a_0 and a_1; that is, a_{k-1} divides m and n.

- Also, $a_i = a_{i-2} - q_i a_{i-1}$ implies that

$$a_i = x a_{i-2} + y a_{i-1} \qquad \text{for some } x, y \in \mathbb{Z}.$$

  for $i = 2, \ldots, k-1$. That is, a_i is a linear combination of a_{i-2} and a_{i-1} where the coefficients are integers. By induction again, we have

$$\begin{aligned} a_{k-1} &= x a_0 + y a_1 \\ a_{k-1} &= x m + y n \qquad \text{for some } x, y \in \mathbb{Z}. \end{aligned}$$

  hence any divisor of m and n is also a divisor of a_{k-1}. Hence a_{k-1} is a gcd.

The Bezout equation

The integers $m, n \in \mathbb{Z}$ are coprime if and only if there exists $x, y \in \mathbb{Z}$ such that

$$xm + yn = 1$$

This equation is called the *Bezout equation*.

Proof

The proof follows immediately from the above proof for Euclid's algorithm.

Notes

- Euclid's algorithm works for
  - The integers $\mathbb{Z}$.
  - Polynomials $\mathbb{R}[s]$.
  - Scalar, stable, proper rational functions in RH_∞.
  - Matrix-valued stable, proper rational functions in RH_∞.
- The general algebraic structure for which this works is called a *ring*.
- The *if* direction is easy; e.g. for polynomials, if m and n have a common zero, then their cannot exist a solution to the Bezout equation.

Scalar stable proper transfer functions

Suppose $m, n \in RH_\infty^{1\times 1}$. Then

d divides n if there exists $q \in RH_\infty^{1\times 1}$ such that $n = dq$

Notes

- d divides n if and only if $\dfrac{n}{d} \in RH_\infty^{1\times 1}$

Examples

- $f_1(s) = \dfrac{s+1}{(s+2)^2}$ $\quad f_2(s) = \dfrac{s-1}{s+1}$ $\quad f_3(s) = \dfrac{s-1}{(s+1)^2}$

  $g_1(s) = \dfrac{s-1}{s+4}$ $\quad g_2(s) = \dfrac{1}{3}$ $\quad g_3(s) = \dfrac{s-1}{(s+2)^2}$

- f_1 divides g_2 and g_3, but not g_1.
- f_2 divides g_1 and g_3, but not g_2.
- f_3 divides g_3, but not g_1 or g_2.

Scalar stable proper transfer functions

$d \in RH_\infty^{1\times 1}$ is called a *greatest common divisor* (gcd) of $n, m \in RH_\infty^{1\times 1}$ if

- d divides n and d divides m.
- Every $a \in RH_\infty^{1\times 1}$ that divides both n and m also divides d.

n and m are called *coprime* if d and d^{-1} are stable and proper for all gcds d.

Notes

- n and m are coprime if and only if they have no common zeros in the right-half-plane, or at infinity.

Examples

- $n = \dfrac{s}{(s+1)^2}$ and $m = \dfrac{s-1}{s+1}$ are coprime. $xm + yn = 1$ is satisfied for

$$x = \frac{(2\,s+4)\,(s+1)^2}{s^3 + 3/2\,s^2 + 3\,s + 1/2} \qquad \text{and} \qquad y = \frac{(s-1/2)\,(s+1)^2}{s^3 + 3/2\,s^2 + 3\,s + 1/2}$$

- $\dfrac{s-1}{(s+3)^2}$ and $\dfrac{s-2}{(s+3)^2}$ are not coprime.

Coprime factorization

Rational numbers

Given $p \in \mathbb{Q}$, find $n, m \in \mathbb{Z}$ such that

$$p = \frac{n}{m} \qquad \text{and } n, m \text{ are coprime}$$

Rational functions; factorization over $\mathbb{R}[s]$

Given $p \in RP^{1\times 1}$ find $n, m \in \mathbb{R}[s]$ such that

$$p = \frac{n}{m} \qquad \text{and } n, m \text{ are coprime}$$

n, m always exist; just cancel any common zeros.

Rational functions; factorization over $RH_\infty^{1\times 1}$

Given $p \in RP^{1\times 1}$ find $n, m \in RH_\infty^{1\times 1}$ such that

$$p = \frac{n}{m} \qquad \text{and } n, m \text{ are coprime}$$

In contrast to above: n, m must be stable proper transfer functions.

Coprime factorization over $RH_\infty^{1\times 1}$

Given $p \in RP^{1\times 1}$ find $n, m \in RH_\infty^{1\times 1}$ such that

$$p = \frac{n}{m} \qquad \text{and } n, m \text{ are coprime}$$

Notes

- n, m must be stable proper transfer functions.
- A coprime factorization always exists; make all stable poles of p poles of n, all stable zeros of p poles of m, and add zeros to n and m as necessary.

Example

Suppose $\hat{p}$ is

$$\hat{p}(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

A coprime-factorization is

$$\hat{n}(s) = \frac{s-1}{s+4} \qquad \hat{m}(s) = \frac{s-3}{s+2}$$

Coprime transfer functions in RH_∞.

Suppose $M, N \in RH_\infty$, and let $D \in RH_\infty$ be square. Then

D right-divides N if there exists $Q \in RH_\infty$ such that $N = QD$

The square $D \in RH_\infty$ us called a *right greatest common divisor* of M, N if

- D right-divides N and D right-divides M.
- Every $A \in RH_\infty^{1\times 1}$ that right-divides both N and M also right-divides D.

N and M are called *right-coprime* if D and D^{-1} are stable and proper for all gcds D.

The Bezout equation

$M, N \in RH_\infty$ are right-coprime if and only if there exists $X, Y \in RH_\infty$ such that

$$XM + YN = I$$

Coprime factorization in RH_∞.

Right-coprime factorization

Given $P \in RP$, a factorization such that

- $P = NM^{-1}$
- $N, M \in RH_\infty$
- N and M are right-coprime

is called a *right-coprime factorization* of P.

Left-coprime factorization

Given $P \in RH_\infty$, a factorization such that

- $P = \tilde{M}^{-1}\tilde{N}$
- $\tilde{N}, \tilde{M} \in RH_\infty$
- $\tilde{N}$ and $\tilde{M}$ are left-coprime

is called a *left-coprime factorization* of P.

Notes

- Left and right coprime factorizations always exist.

Example

Suppose P is

$$\hat{P}(s) = \frac{s}{(s+1)(s-1)}$$

A coprime-factorization is

$$N(s) = \frac{s}{(s+1)^2} \qquad M(s) = \frac{s-1}{s+1}$$

Stabilization via coprime factorization

Scalar example

Suppose $\hat{p}_{22} \in RH_\infty^{1\times 1}$. Let

$$\hat{p}_{22}(s) = \frac{\hat{n}(s)}{\hat{m}(s)}$$

be a coprime factorization, and $\hat{x}, \hat{y} \in RH_\infty^{1\times 1}$ satisfy the Bezout equation

$$\hat{x}(s)\hat{m}(s) - \hat{y}(s)\hat{n}(s) = 1$$

Theorem

$\hat{k}(s) = \dfrac{\hat{y}(s)}{\hat{x}(s)}$ is a stabilizing controller.

Proof

$$\begin{aligned} \hat{Z} = \begin{bmatrix} I & -\hat{k} \\ -\hat{p}_{22} & I \end{bmatrix}^{-1} &= \frac{1}{1-\hat{k}\hat{p}_{22}} \begin{bmatrix} 1 & \hat{k} \\ \hat{p}_{22} & 1 \end{bmatrix} \\ &= \frac{1}{\hat{x}\hat{m} - \hat{y}\hat{n}} \begin{bmatrix} \hat{x}\hat{m} & \hat{y}\hat{m} \\ \hat{x}\hat{n} & \hat{x}\hat{m} \end{bmatrix} = \begin{bmatrix} \hat{x}\hat{m} & \hat{y}\hat{m} \\ \hat{x}\hat{n} & \hat{x}\hat{m} \end{bmatrix} \end{aligned}$$

which is stable.

Every stabilizing controller

Suppose $\hat{p}_{22} \in RH_\infty^{1\times 1}$. Let $\hat{p}_{22}(s) = \hat{n}(s)\hat{m}^{-1}(s)$ be a coprime factorization, and $\hat{x}, \hat{y} \in RH_\infty^{1\times 1}$ satisfy the Bezout equation $\hat{x}(s)\hat{m}(s) - \hat{y}(s)\hat{n}(s) = 1$.

Theorem

Every stabilizing controller has the form

$$\hat{k} = \frac{\hat{y} - \hat{m}\hat{q}}{\hat{x} - \hat{n}\hat{q}}$$

for some $q \in RH_\infty^{1\times 1}$.

Proof

The proof that $\hat{k}$ is stabilizing is the same as before, since

$$(\hat{x} - \hat{n}\hat{q})\hat{m} - (\hat{y} - \hat{m}\hat{q}) = 1$$

Then

$$\hat{Z} = \begin{bmatrix} (\hat{x} - \hat{n}\hat{q})\hat{m} & (\hat{y} - \hat{m}\hat{q})\hat{m} \\ (\hat{x} - \hat{n}\hat{q})\hat{n} & (\hat{x} - \hat{n}\hat{q})\hat{m} \end{bmatrix}$$

which is stable.

We will prove that every $\hat{k}$ has this form in the matrix case.