Engr210a Lecture 15: State-space computations

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Formulae for coprime factorization

Suppose we have the state-space system G

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) = 0$$
$$y(t) = Cx(t) + Du(t)$$

Choose F such that $A + BF$ is Hurwitz, and L such that $A + LC$ is Hurwitz. Then the a doubly coprime factorization is given by

$$\begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & -L \\ \hline F & I & 0 \\ C + DF & D & I \end{array}\right]$$

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -B - LD & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

Proof

One can prove this by direct multiplication.

Stabilization via LMIs

If $Q > 0$, then a matrix A is Hurwitz if and only if the solution to

$$AX + XA^* + Q = 0$$

satisfies $X > 0$.

Equivalently, A is Hurwitz if and only if there exists $X > 0$ such that

$$AX + XA^* < 0$$

To find F such that $A + BF$ is Hurwitz, first note that

$$A + BF \text{ is Hurwitz} \qquad \Longleftrightarrow \qquad \exists\, X > 0 \; ; \; (A + BF)X + X(A + BF)^* < 0$$

This is not an LMI in F and X, since the product FX appears. Substitute $Z = FX$, then we have the following.

Theorem

There exists F such that $A + BF$ is Hurwitz if and only if there exists $X > 0$ and Z such that

$$AX + XA^* + BZ + Z^*B^* < 0$$

Then one such F is $F = ZX^{-1}$.

State-space description of stabilizing controllers

Suppose (A, B_2) is stabilizable and (A, C_2) is detectable. Let F and L be matrices such that $A + B_2F$ and $A + LC_2$ are Hurwitz. Then

$$\hat{K} = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right]$$

is a stabilizing controller.

Proof

The dynamics of the interconnected system are $\begin{bmatrix} \dot{x}(t) \\ \dot{x}_K(t) \end{bmatrix} = A_{\mathsf{cl}} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix}$ where

$$A_{\mathsf{cl}} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

Substituting the controller parameters from above gives

$$A_{\mathsf{cl}} = \begin{bmatrix} A & B_2F \\ -LC_2 & A + LC_2 + B_2F \end{bmatrix} = T^{-1} \begin{bmatrix} A + LC_2 & 0 \\ -LC_2 & A + B_2F \end{bmatrix} T$$

for $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$. Hence the eigenvalues of A_{cl} are those of $A + B_2F$ and $A + LC_2$.

The Kalman-Yakubovich-Popov Lemma

Suppose we have the state-space system G

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(t) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Then the following are equivalent

- $\|G\| < 1$ and A is Hurwitz
- There exists $X \in \mathbb{R}^{n\times n}$, with $X = X^*$ and $X > 0$ such that

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} + \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Notes

- Also called the *Bounded Real Lemma*.
- The above condition is an LMI in X; solvable by semidefinite programming.
- The $(1,1)$ block is $A^*X + XA + C^*C < 0$ is a Lyapunov inequality. Existence of $X > 0$ satisfying this LMI implies A is Hurwitz.
- The $(2,2)$ block is $D^*D < I$; necessary for $\|G\| < 1$.

Dissipativity

Suppose we have the nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$.

Suppose we have a function

$$\begin{aligned} s : \mathbb{R}^m \times \mathbb{R}^p &\to \mathbb{R} \\ (u(t), y(t)) &\mapsto s(u(t), y(t)) \end{aligned}$$

We call s the *supply rate* function.

Dissipativity

The system is called *dissipative* if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(z) \geq 0$ for all $z \in \mathbb{R}^n$
- $V(z_1) + \int_0^{t_1} s\big(u(t), y(t)\big)\, dt \leq V(z_0)$ for every u, y, z_1, z_0 satisfying

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \qquad x(0) = z_0 \\ y(t) &= g(x(t), u(t)) \\ z_1 &= x(t_1) \end{aligned}$$

  for every $t_1 \geq 0$.

Note: We need smoothness conditions on u, f, g to make this precise.

Dissipation inequality

If V and x are sufficiently smooth, this is equivalent to

$$\frac{d}{dt} V\big(x(t)\big) \leq -s\big(u(t), y(t)\big)$$

where u, y, x are related by the dynamics of the system.

Interpretation of dissipativity

$$\frac{d}{dt}V\big(x(t)\big) \leq -s\big(u(t), y(t)\big)$$

- V is the amount of *substance or energy* in the system
- $s(u(t), y(t))$ is the rate at which substance or energy is obtained from the system.

Examples

- Electrical systems: $u(t)$ is a vector of voltages, $y(t)$ is a vector of currents, and the supply function is $s\big(u(t), y(t)\big) = -u(t)^* y(t)$.
- Mechanical systems: $u(t)$ is a vector of forces, $y(t)$ is a vector of velocities, and the supply function is $s\big(u(t), y(t)\big) = -u(t)^* y(t)$.

Dissipation inequality

- The dissipation inequality can be written

$$\frac{\partial V(x,u)}{\partial x} f(x,u) + s\big(u, g(x,u)\big) \leq 0 \qquad \text{for all } x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

- Called *Hamilton-Jacobi* equation or *Bellman* equation.

Connection to Lyapunov theory

Suppose $f(0,0) = 0$ and $s(0, g(z,0)) \geq 0$ for all $z \in \mathbb{R}^n$. Then the dissipation inequality implies

$$\frac{\partial V(x,0)}{\partial x} f(x,0) \leq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Lyapunov functions

Recall that if $V : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(x) > 0$ for $x \neq 0$

(iii) $\displaystyle \frac{d}{dt} V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0$ for $x \neq 0$.

(iv) If $\{x_0, x_1, \ldots\}$ is a sequence such that $\|x_k\| \to \infty$, then $V(x_k) \to \infty$.

So, provided V satisfies condition (iv), the origin $x = 0$ is globally asymptotically stable with zero input. That is, for any initial condition

$$\lim_{t \to \infty} x(t) = 0$$

Theorem

The system is dissipative if and only if for every z_0, there exists $c > 0$ such that

$$\int_0^{t_1} s\big(u(t), y(t)\big)\, dt < c \qquad \text{for all } t_1 \geq 0 \text{ and all } u.$$

where $x(0) = z_0$, and x, y are functions of z_0, u determined by the system dynamics.

Proof

(only if:) If the system is dissipative, then

$$V(z_1) - V(z_0) \leq -\int_0^{t_1} s\big(u(t), y(t)\big)\, dt \qquad \text{for all } t_1 \geq 0 \text{ and all } u.$$

where $x(0) = z_0$, and x, y and $z_1 = x(t_1)$ are functions of z_0, u determined by the system dynamics.

This implies that for any $z_0 \in \mathbb{R}^n$,

$$\int_0^{t_1} s\big(u(t), y(t)\big)\, dt \leq V(z_0) \qquad \text{for all } t_1 \geq 0 \text{ and all } u.$$

Proof (if)

- We will show that if $\int_0^{t_1} s\big(u(t), y(t)\big)\, dt < c$ for all $t_1 \geq 0$ and all u, then the system is dissipative.

- Let $V(z) = \sup\left\{ \int_0^{\tau} s\big(u(t), y(t)\big)\, dt \;;\; t_1 \geq 0, u \text{ on } [0, t_1], x(0) = z \right\}$
where x, y are functions of z, u determined by the system dynamics.

- Clearly $V(z) \geq 0$ for all $z \in \mathbb{R}^n$.

- $$V(z_0) \geq \sup_{u|_{[0,t_2]}} \int_0^{t_2} s\big(u(t), y(t)\big)\, dt \qquad \text{for all } t_2 \geq 0$$
$$= \sup_{u|_{[0,t_1]}} \left\{ \int_0^{t_1} s\big(u(t), y(t)\big)\, dt + \sup_{u|_{[t_1,t_2]}} \int_{t_1}^{t_2} s\big(u(t), y(t)\big)\, dt \right\} \qquad \text{for all } t_2 \geq 0$$

- Hence $V(z_0) \geq \int_0^{t_1} s\big(u(t), y(t)\big)\, dt + V(z_1)$ for all u on $[0, t_1]$.

- This approach is known as the *Bellman principle* or *dynamic programming*.

The induced-norm via dissipativity

Consider the linear system G

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(t) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Pick supply function as the quadratic function

$$s\big(u(t), y(t)\big) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

Then

$$\begin{aligned} \int_0^\infty s\big(u(t), y(t)\big)\, dt &= \int_0^\infty \Big(y(t)^* y(t) - u(t)^* u(t) \Big)\, dt \\ &= \|y\|^2 - \|u\|^2 \end{aligned}$$

Recall that

$$\begin{aligned} \|G\| \leq 1 \quad &\iff \quad \|y\|^2 \leq \|u\|^2 \quad \text{for all } u \in L_2[0, \infty) \\ &\iff \quad \|y\|^2 - \|u\|^2 \leq 0 \quad \text{for all } u \in L_2[0, \infty) \end{aligned}$$

Theorem

The system satisfies $\|G\| \leq 1$, where G is the map $u \mapsto y$ with initial condition $x(0) = 0$, if and only if for every z_0, there exists $c > 0$ such that

$$\int_0^{t_1} s\big(u(t), y(t)\big)\, dt < c \qquad \text{for all } t_1 \geq 0 \text{ and all } u.$$

where $x(0) = z_0$, and x, y are functions of z_0, u determined by the system dynamics, and s is the above defined quadratic storage function.

Notes

- With this s, the system is dissipative if and only if $\|G\| \leq 1$.

Proof $\Longleftarrow$

Suppose $\|G\| > 1$. Then there exists u_0 such that $\|Gu_0\|^2 > \|u_0\|^2$ with $x(0) = 0$. That is, there exists $c > 0$ such that $\|Gu_0\|^2 - \|u_0\|^2 > c$. By scaling u_0, we can make $\|Gu_0\|^2 - \|u_0\|^2$ arbitrarily large.

Proof $\implies$

- We wish to show that for every z_0, there exists $c > 0$ such that

$$\int_0^{t_1} s\big(u(t), y(t)\big)\, dt < c \qquad \text{for all } t_1 \geq 0 \text{ and all } u.$$

where $x(0) = z_0$, and x, y are functions of z_0, u determined by the system dynamics.

- Suppose $\|G\| \leq 1$. Then when $x(0) = 0$,

$$\|y\|^2 - \|u\|^2 \leq 0 \qquad \text{for all } u \in L_2[0, \infty)$$

- Hence, for any z_0, when $x(0) = z_0$ we have

$$\|y\|^2 - \|u\|^2 \leq \|y_{\text{free}}\|^2 \qquad \text{for all } u \in L_2[0, \infty)$$

where $y_{\text{free}}(t) = Ce^{At}z_0$. Hence

$$\int_0^{\infty} s\big(u(t), y(t)\big)\, dt \leq \|y_{\text{free}}\|^2 \qquad \text{for all } u \in L_2[0, \infty)$$

- Suppose there exists t_1 such that $\displaystyle\int_0^{t_1} s\big(u(t), y(t)\big)\, dt$ can be made arbitrarily large.

Then we can set $u(t) = 0$ on $t > t_1$, and contradict the above statement.

Linear systems and dissipativity

Consider the linear system G

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t) = 0$$
$$y(t) = Cx(t) + Du(t)$$

Pick supply function as the quadratic function $s\big(u(t), y(t)\big) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$

For dissipative LTI systems with quadratic supply functions, we can always find quadratic storage functions, $V(x) = x^* X x$. The dissipation inequality is then

$$\frac{\partial V(x,u)}{\partial x} f(x,u) + s\big(u, g(x,u)\big) \leq 0 \qquad \text{for all } x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

which holds if and only if

$$(Ax + Bu)^* X x + x^* X (Ax + Bu) + \begin{bmatrix} u \\ y \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \leq 0$$

holds for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

The induced-norm and the dissipation inequality

The system is dissipative if and only if, for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$,

$$(Ax+Bu)^*Xx + x^*X(Ax+Bu) + \begin{bmatrix} u \\ y \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \leq 0$$

which holds if and only if, for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$,

$$\begin{bmatrix} x \\ u \end{bmatrix}^* \left(\begin{bmatrix} XA & XB \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^*X & 0 \\ B^*X & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix}^* \leq 0$$

which holds if and only if

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^* \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \leq 0$$

For the induced-norm, we need $P = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$, which gives

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} + \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0$$

This is the KYP LMI. Note $X > 0$ if V is a storage function.

Riccati inequality

We have $\|G\| < 1$ if and only if there exists $X \in \mathbb{R}^{n\times n}$, with $X = X^*$ and $X > 0$ such that

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} + \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

This is equivalent to

$$\begin{bmatrix} A^*X + XA + C^*C & XB + C^*D \\ B^*X + D^*C & -I + D^*D \end{bmatrix} < 0$$

Taking the Schur complement, this holds if and only if

$$A^*X + XA + C^*C + (XB + C^*D)(I - D^*D)^{-1}(B^*X + D^*C) < 0$$

$$\text{and} \qquad D^*D - I < 0$$

This is called a *Riccati inequality*.

When $D = 0$, it becomes

$$A^*X + XA + C^*C + XBB^*X < 0$$

Riccati equation

The KYP lemma may also be stated as the following. The norm $\|G\| < 1$ if and only if $\|D\| < 1$ and there exists $X = X^*$, such that

$$A^*X + XA + C^*C + (XB + C^*D)(I - D^*D)^{-1}(B^*X + D^*C) = 0$$

where $A + B(I - D^*D)^{-1}(B^*X + D^*C)$ is Hurwitz.

Notes

- This is the Riccati equation form.
- When such an X exists, it satisfies $X > 0$.