Engr210a Lecture 17: LFTs and robustness

- Additive perturbations
- General problem formulation
- Example of parametric uncertainty
- Small-gain theorem
- Interconnections
- Robust performance
- Linking robust performance and robust stability
- Diagonal perturbations
- Scaling
- Necessity

Additive perturbations

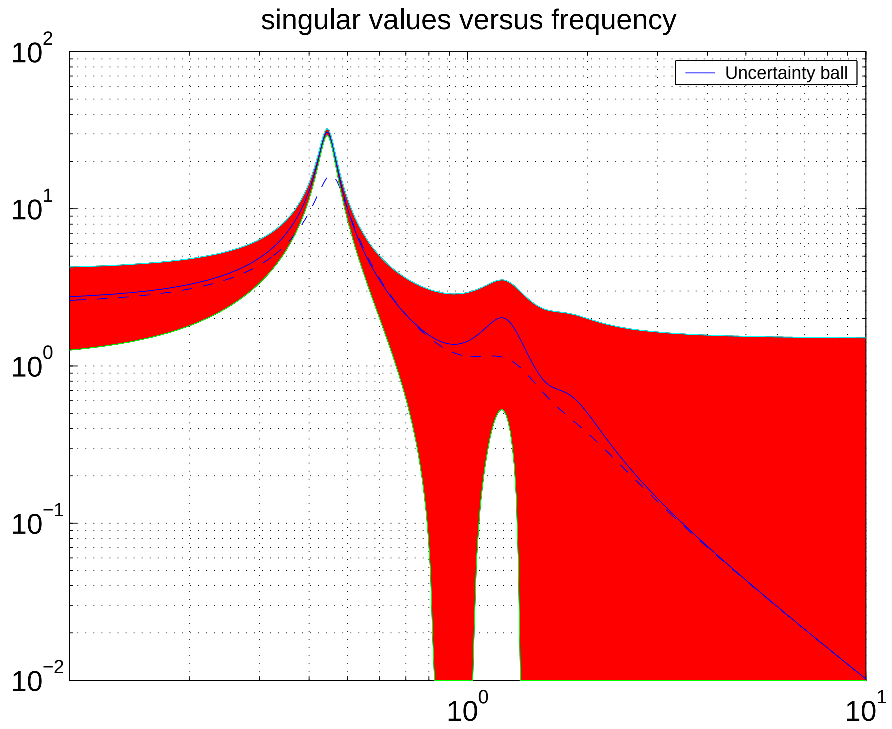
Instead of trying to design a control system for G_{n}, try to design a controller that achieves a specified level of performance for any G such that

$$\|G - G_{\mathsf{nominal}}\| < c$$

In other words, design a controller that will work for any G such that

$$G = G_{\mathsf{nominal}} + \Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

This sounds reasonable, but leads to large uncertainty at small values of $\hat{G}(j\omega)$.

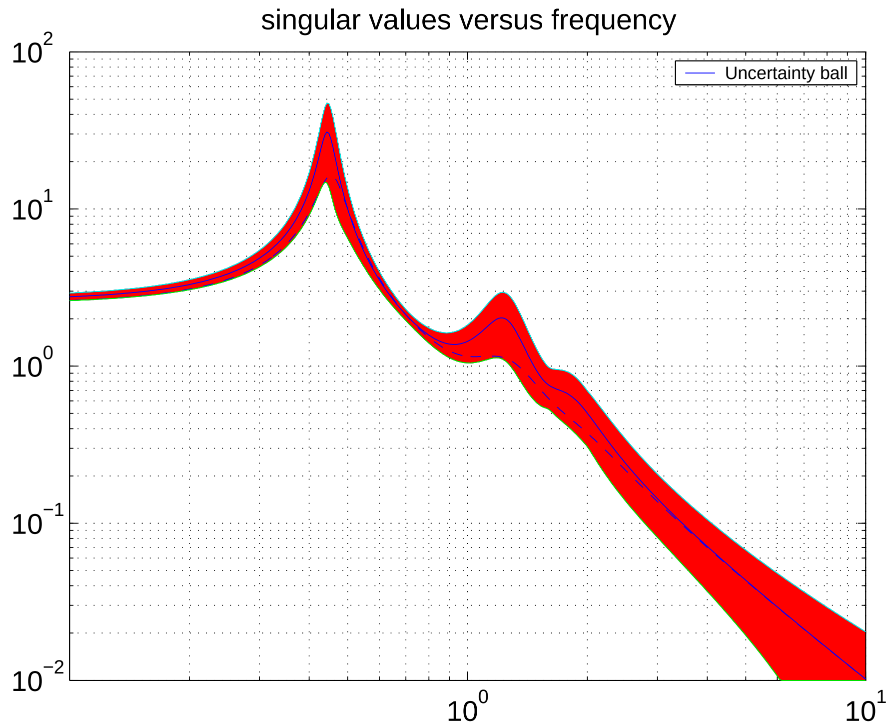


Weighted additive uncertainty

Design a controller that achieves a specified level of performance for any G such that

$$G = G_{\text{nominal}} + W\Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

Here W is a transfer function, chosen to be small at frequencies where the model is good, and large elsewhere.

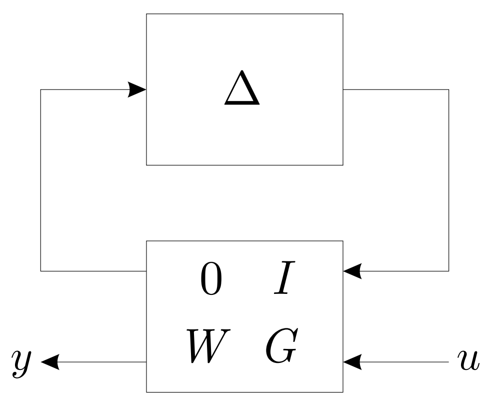


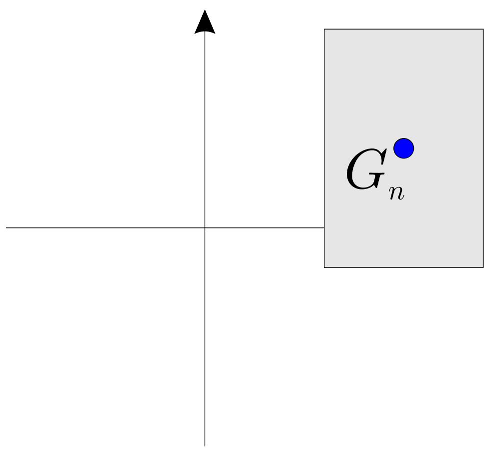
Weighted additive uncertainty

Design a controller that achieves a specified level of performance for any G such that

$$G = G_{\mathsf{nominal}} + W\Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

We are therefore trying to do a control design for a *set of systems*, not just a single system. This particular set is a *ball* in H_∞. It is called a *weighted additive uncertainty ball*.



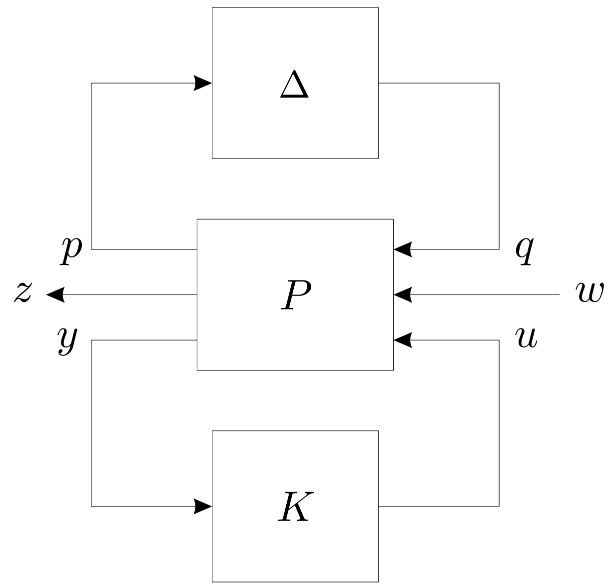


We can also represent this as the above *linear-fractional transformation*.

Here the system $G = \begin{bmatrix} 0 & I \\ W & G \end{bmatrix}$ is called *the generalized plant*.

General problem setup

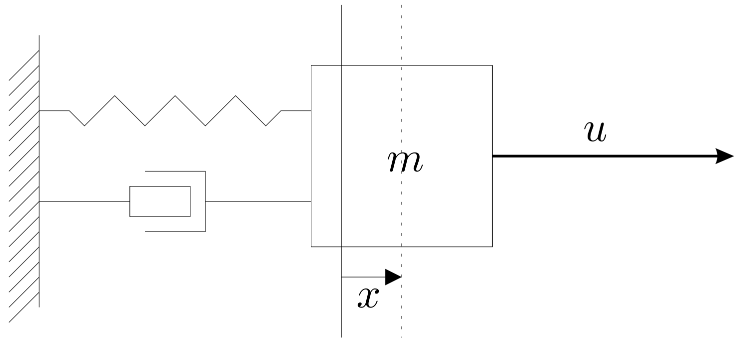
We will consider the general problem



Interpretation

- Δ is the *model uncertainty*.
- z is a signal we would like to keep small
- w represents external disturbances

Example



The equation of motion is

$$\ddot{x}(t) + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{u}{m}$$

Parametric uncertainty

- Suppose we know m within 10%, c within 20%, and k within 30%.

$$m = m_n(1 + 0.1\delta_m)$$
$$c = c_n(1 + 0.2\delta_c)$$
$$k = k_n(1 + 0.3\delta_k)$$

- Here $|\delta_m| \le 1$, $|\delta_k| \le 1$, $|\delta_c| \le 1$.

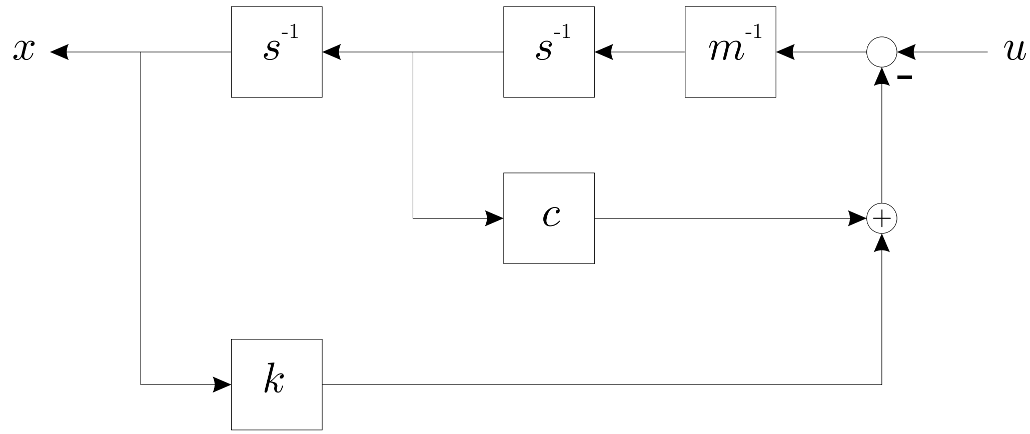
- m_n is called the *nominal value* of m, and δ_m is called the *perturbation*.

Block-diagram

The equation of motion is

$$\ddot{x}(t) + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{u}{m}$$

In block diagram form



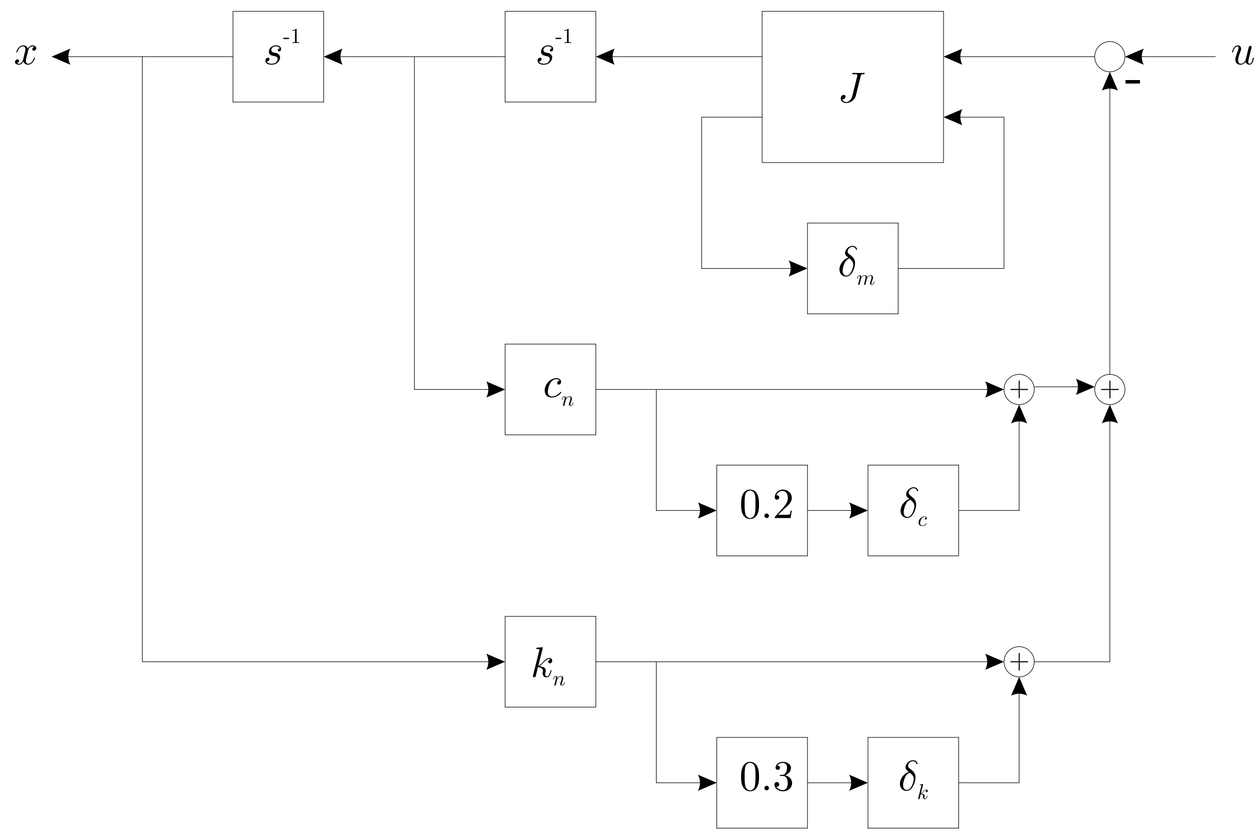
Easy to verify that $\frac{1}{m} = \frac{1}{m_n(1+0.1\delta_m)}$ where $J = \begin{bmatrix} m_n^{-1} & -0.1m_n^{-1} \\ 1 & -0.1 \end{bmatrix}$

$$= {m_n}^{-1} - 0.1{m_n}^{-1}\delta_m(1+0.1\delta_m)^{-1}$$

$$= \underline{S}(J, \delta_m)$$

Block-diagram

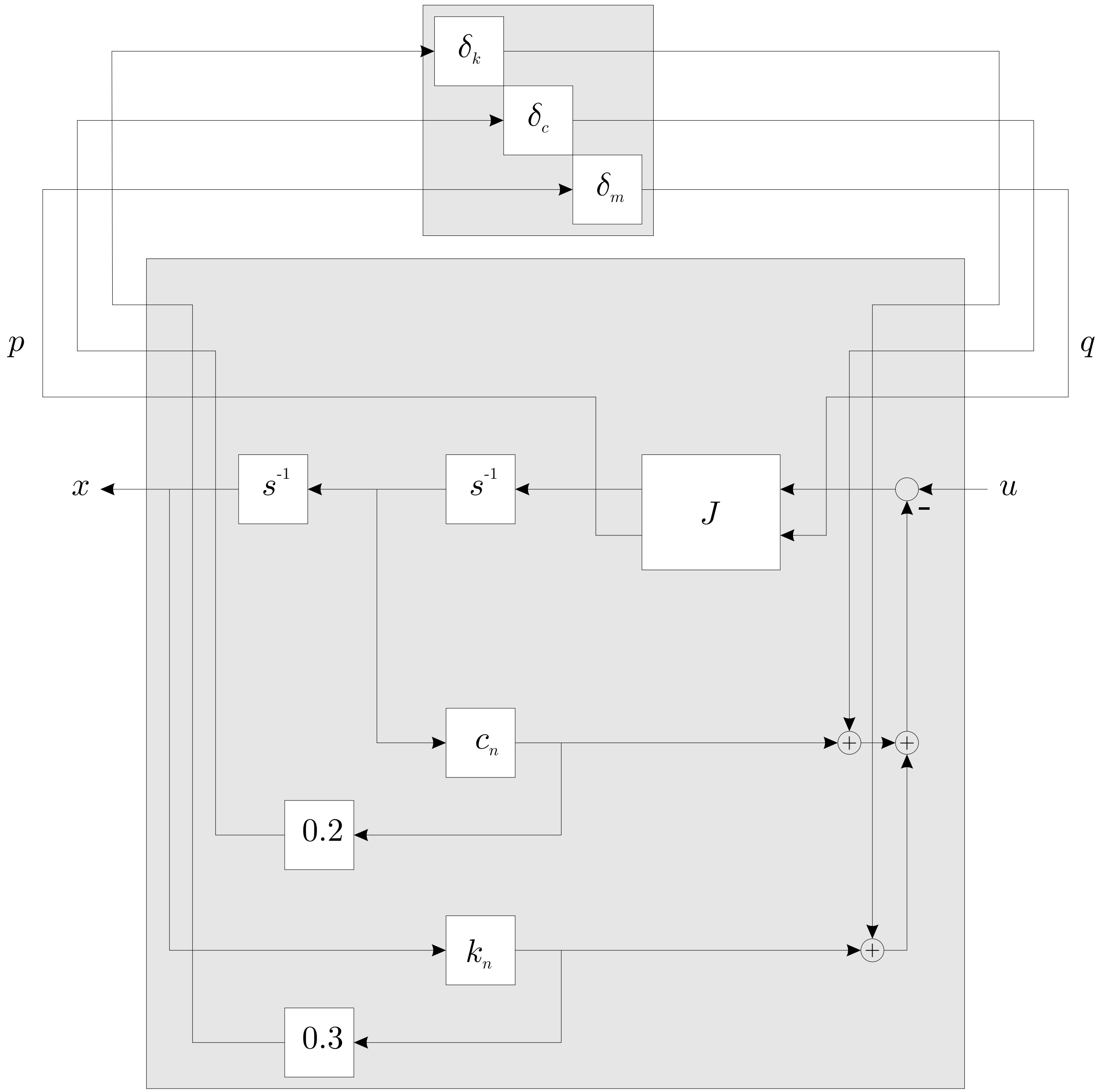
With the perturbations we have



where

$$J = \begin{bmatrix} m_n^{-1} & -0.1 m_n^{-1} \\ 1 & -0.1 \end{bmatrix}$$

Block-diagram



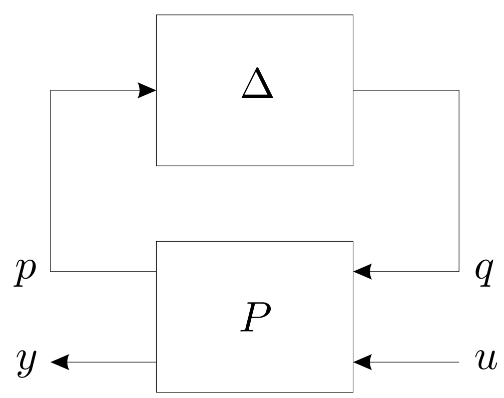
State-space form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k_n m_n^{-1} & -c_n m_n^{-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ -m_n^{-1} & -m_n^{-1} & -0.1 m_n^{-1}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ m_n^{-1}\end{bmatrix} u$$

$$\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix} = \begin{bmatrix}0.3k_n & 0\\ 0 & 0.2c_n\\ -k_n & -c_n\end{bmatrix} x + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & -1 & -0.1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u$$

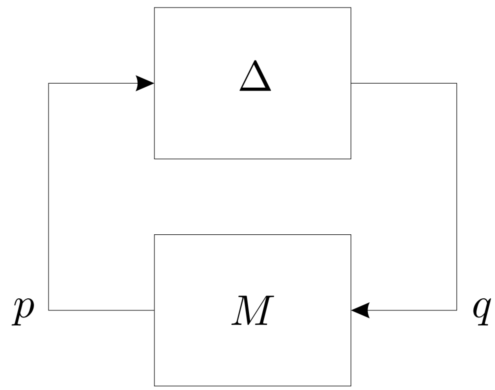
$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

LFT representation



- Note Δ is block-diagonal.

Small-gain theorem, version 2



Assumptions

- $M \in \mathcal{L}(L_2)$.
- $\Delta \in \mathcal{L}(L_2)$.

Theorem

The closed-loop is input-output stable for all Δ such that $\|\Delta\| \leq 1$ if and only if $\|M\| < 1$.

Small-gain theorem, version 2

The closed-loop is input-output stable for all Δ such that $\|\Delta\| \leq 1$ if and only if $\|M\| < 1$.

Proof

Recall the closed-loop is stable if and only if

$$Z = \begin{bmatrix} I & -\Delta \\ -M & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - \Delta M)^{-1} & \Delta(I - M\Delta)^{-1} \\ (I - M\Delta)^{-1}M & (I - M\Delta)^{-1} \end{bmatrix}$$

is stable.

(if) We know $\|M\Delta\| \leq \|M\|\|\Delta\| < 1$. Hence $I - M\Delta$ is invertible.

(only if) We need to show that

$$\|M\| \geq 1 \quad \Longrightarrow \quad \text{There exists } \Delta,\ \|\Delta\| \leq 1, \text{ such that } I - M\Delta \text{ is singular}$$

For any M, $\rho(MM^*) = \|M\|^2 \geq 1$.

Let $\lambda = \rho(MM^*)$. Then since $\operatorname{spec}(MM^*)$ is closed

$$\lambda I - QQ^* \text{ is singular}$$

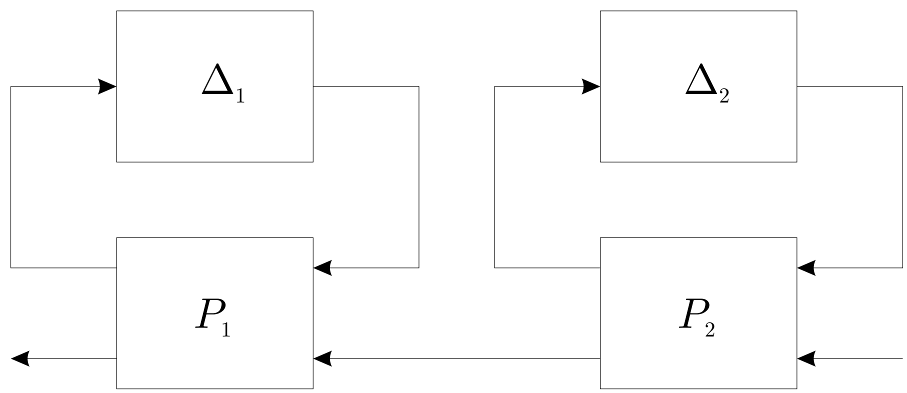
So choose $\Delta = \lambda^{-1}Q^*$.

Interconnecting uncertain systems

Block-diagonal uncertainty arises

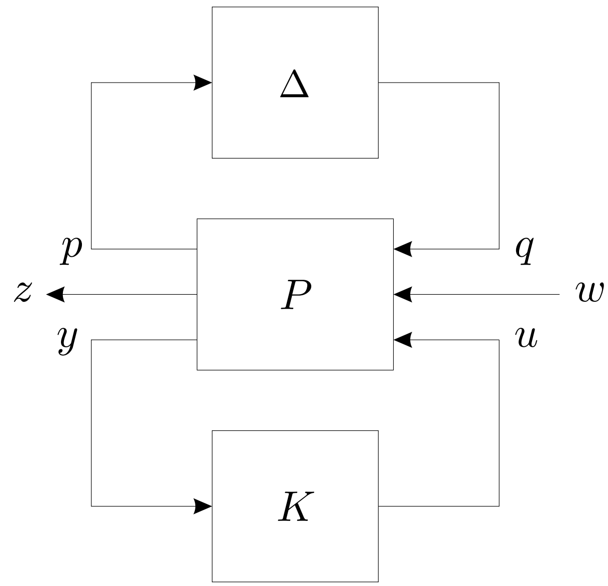
- From uncertain parameters
- From interconnected uncertain subsystems

Example: cascade



This can be written as an LFT on $\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$.

Robust performance



The closed-loop map $T : w \mapsto z$ is a function $T(\Delta, K)$.

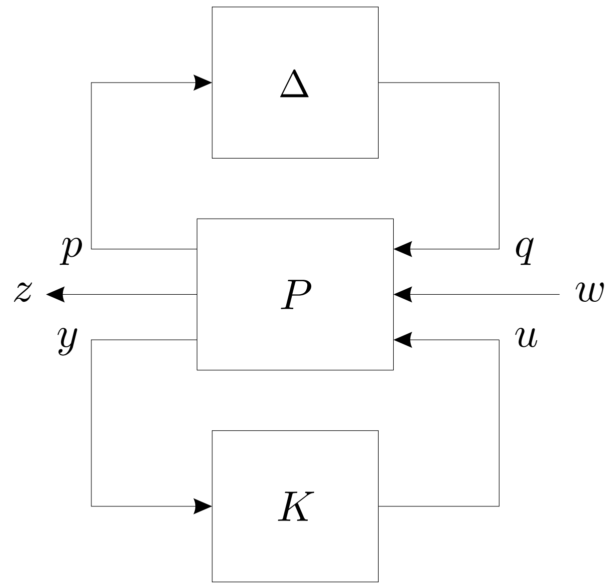
Control objective

Find K to solve

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subject to } \|T(\Delta, K)\| \leq \gamma \text{ for all } \Delta \text{ with } \|\Delta\| \leq 1. \end{aligned}$$

Often we have additional constraints, that Δ be block-diagonal.

Robust performance

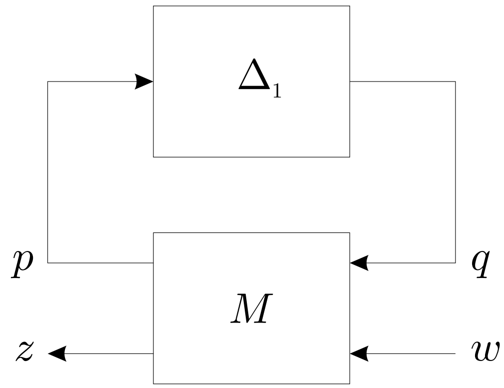


Worst-case interpretation

Find K to solve

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subject to } \max\Big\{ \|T(\Delta, K)\| \;;\; \|\Delta\| \leq 1 \Big\} \leq \gamma \end{aligned}$$

Robust performance and robust stability

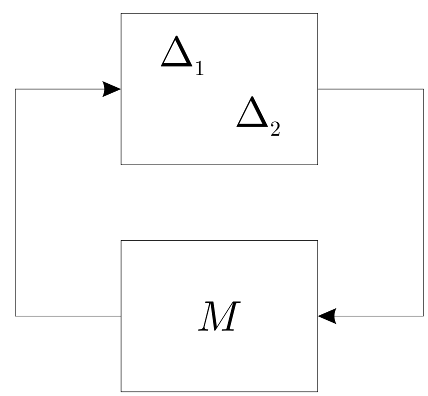


Interconnection

$$z = \overline{S}(M, \Delta_1)w = (M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12})w$$

Theorem

$$\max\left\{\|\overline{S}(M, \Delta_1)\| \ ; \ \|\Delta_1\| \leq 1\right\} < 1 \qquad \Longleftrightarrow$$



is input-output stable

for all Δ, block-diagonal, $\|\Delta\| \leq 1$

Robust performance and robust stability

Define the set

$$\mathbf{\Delta} = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \; ; \; \|\Delta_1\| \leq 1, \|\Delta_2\| \leq 1 \right\}$$

Then the following are equivalent

(i) $I - M_{11}\Delta_1$ is invertible and $\|\overline{S}(M, \Delta_1)\| < 1$ for all Δ_1 with $\|\Delta_1\| \leq 1$.

(ii) $I - M\Delta$ is invertible for all Δ with $\|\Delta\| \leq 1$.

Notes

- (i) is a *robust performance* specification
- (ii) is a *robust stability* specification.

Proof

We want to prove that the following are equivalent

(i) $I - M_{11}\Delta_1$ is invertible and $\|\overline{S}(M, \Delta_1)\| < 1$ for all Δ_1 with $\|\Delta_1\| \leq 1$.

(ii) $I - M\Delta$ is invertible for all Δ with $\|\Delta\| \leq 1$.

First we show (i) $\implies$ (ii)

- We know

$$I - M\Delta = I - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} = \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ -M_{21}\Delta_1 & I - M_{22}\Delta_2 \end{bmatrix}$$

- Hence

$$I - M\Delta = \begin{bmatrix} I & 0 \\ -M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1} & I \end{bmatrix} \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ 0 & I - \bar{S}(M, \Delta_1)\Delta_2 \end{bmatrix}$$

  Hence $I - M\Delta$ is nonsingular if $I - \bar{S}(M, \Delta_1)\Delta_2$ is nonsingular.

- This follows by assumption that $\|\bar{S}(M, \Delta_1)\| < 1$.

Proof

We want to prove that the following are equivalent

(i) $I - M_{11}\Delta_1$ is invertible and $\|\overline{S}(M,\Delta_1)\| < 1$ for all Δ_1 with $\|\Delta_1\| \leq 1$.

(ii) $I - M\Delta$ is invertible for all Δ with $\|\Delta\| \leq 1$.

We now show (ii) $\implies$ (i)

- Choose $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $\|\Delta_1\| \leq 1$. Then

$$I - M\Delta = \begin{bmatrix} I - M_{11}\Delta_1 & 0 \\ -M_{21}\Delta_1 & I \end{bmatrix} \text{ is nonsingular}$$

  by assumption, hence $I - M_{11}\Delta_1$ is nonsingular for all Δ_1 with $\|\Delta_1\| \leq 1$.

- For all Δ with $\|\Delta\| \leq 1$, we have

$$I - M\Delta = \begin{bmatrix} I & 0 \\ -M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1} & I \end{bmatrix} \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ 0 & I - \bar{S}(M,\Delta_1)\Delta_2 \end{bmatrix}$$

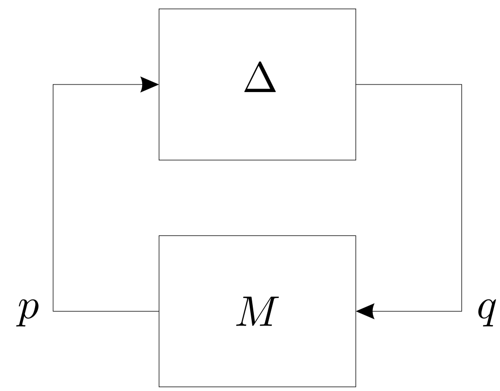
  is nonsingular.

- Hence by the small gain theorem, $\|\bar{S}(M,\Delta_1)\| < 1$ for all Δ_1 with $\|\Delta_1\| \leq 1$.

Diagonal perturbations and the small-gain theorem

We are interested in diagonal perturbations of the form

$$\boldsymbol{\Delta} = \Big\{ \mathrm{diag}(\Delta_1, \dots, \Delta_d) \ ; \ \Delta_i \in \mathcal{L}(L_2), \ \|\Delta_i\| \le 1 \Big\}$$



Notes

- $\|M\| < 1$ if and only if the closed-loop is stable for all $\|\Delta\| \le 1$.
- But we have a limited class of Δ; those in $\boldsymbol{\Delta}$.
- Clearly $\|M\| < 1$ implies stability.
- What about necessity?

Scaling the small-gain theorem

Diagonal perturbations

$$\mathbf{\Delta} = \Big\{ \mathrm{diag}(\Delta_1, \ldots, \Delta_d) \ ; \ \Delta_i \in \mathcal{L}(L_2), \ \|\Delta_i\| \leq 1 \Big\}$$

Define the set of operators

$$\mathbf{\Theta} = \Big\{ \Theta \in \mathcal{L}(L_2), \ \Theta \text{ is invertible}, \ \Theta\Delta = \Delta\Theta \text{ for all } \Delta \in \mathbf{\Delta} \Big\}$$

This set is called the *commutant* of $\mathbf{\Delta}$.

Notes

- If Θ commutes with Δ, then so does Θ^{-1}.
- We have

$$\begin{aligned} & I - M\Delta \text{ is invertible} \\ \iff \quad & I - \Theta^{-1}\Theta M\Delta \text{ is invertible} \\ \iff \quad & I - \Theta M\Delta\Theta^{-1} \text{ is invertible} \\ \iff \quad & I - \Theta M\Theta^{-1}\Delta \text{ is invertible} \end{aligned}$$

- *Scaled small-gain test:* Robust stability if $\|\Theta M \Theta^{-1}\| < 1$ for some $\Theta \in \mathbf{\Theta}$.

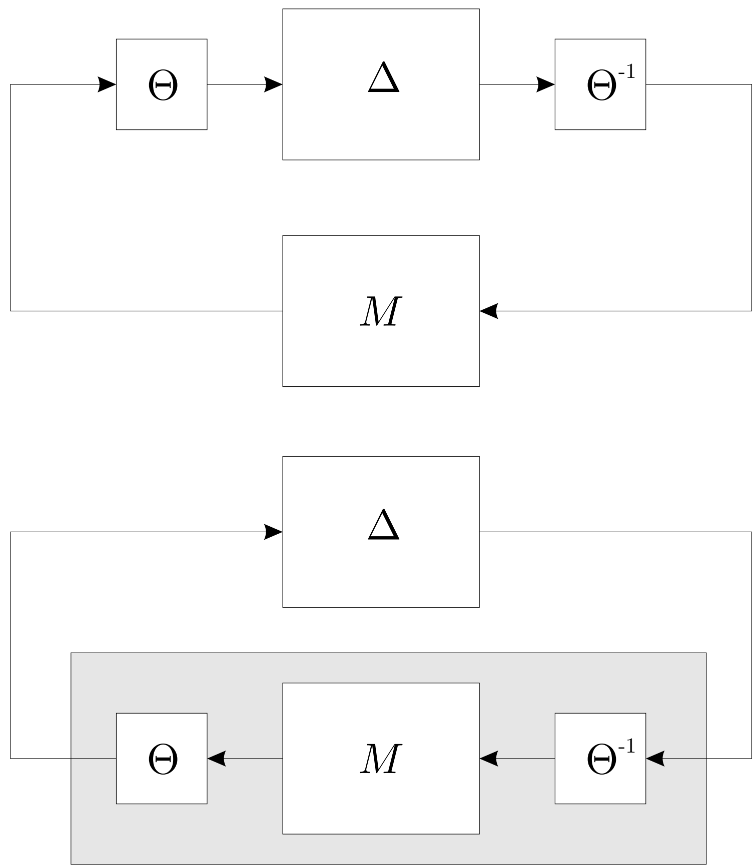
Scaled small-gain theorem

Suppose there exists $\Theta \in \mathbf{\Theta}$ such that

$$\|\Theta M \Theta^{-1}\| < 1$$

then the closed-loop is robustly input-output stable.

Feedback interpretation



The commutant set

For diagonal perturbations

$$\mathbf{\Delta} = \Big\{ \operatorname{diag}(\Delta_1, \ldots, \Delta_d) \; ; \; \Delta_i \in \mathcal{L}(L_2), \; \|\Delta_i\| \leq 1 \Big\}$$

The commutant set is

$$\mathbf{\Theta} = \Big\{ \operatorname{diag}(\theta_1 I, \ldots, \theta_d I) \; ; \; \theta_i \in \mathbb{C} \Big\}$$

Notes

- If we allow $\mathbf{\Delta}$ to contain arbitrary operators Δ_i, then the commutant set consists of diagonal matrices.
- Other sets of operators have other commutant sets; for example, time-invariant operators.

Scaled small-gain computation

Define the set

$$\mathbf{P\Theta} = \left\{ \operatorname{diag}(\theta_1 I, \ldots, \theta_d I) \;;\; \theta_i > 0 \right\}$$

Theorem

The following are equivalent

(i) There exist $\Theta \in \mathbf{\Theta}$ such that

$$\|\Theta M \Theta^{-1}\| < 1$$

(ii) There exist $\Theta \in \mathbf{P\Theta}$ such that

$$M^* \Theta M - \Theta < 0$$

Scaled small-gain computation

The following are equivalent

(i) There exist $\Theta \in \mathbf{\Theta}$ such that

$$\|\Theta M \Theta^{-1}\| < 1$$

(ii) There exist $\Theta \in \mathbf{P}$ and $X > 0$ such that

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & -\Theta \end{bmatrix} + \begin{bmatrix} C^* \\ D^* \end{bmatrix} \Theta \begin{bmatrix} C & D \end{bmatrix} < 0$$

Proof

Follows from KYP lemma applied to

$$\Theta^{\frac{1}{2}} \hat{M} \Theta^{-\frac{1}{2}} = \left[\begin{array}{c|c} A & B\Theta^{-\frac{1}{2}} \\ \hline \Theta^{\frac{1}{2}} C & \Theta^{\frac{1}{2}} D \Theta^{-\frac{1}{2}} \end{array}\right]$$

Scaled small-gain computation

So far

If there exists $\Theta \in \boldsymbol{\Theta}$ such that $\|\Theta M \Theta^{-1}\| < 1$, then the closed-loop is robustly stable.

Necessity

- Major question: is this condition necessary?
- Equivalently: if there does not exist such a Θ, is the system *not* robustly stable?

Major result:

- For arbitrary operators Δ_i, this condition is necessary.
- For more restrictive classes, such as LTI perturbations and scalar parameters, the condition is *not* necessary.