Engr210a Lecture 6: Linear analysis and systems

- Banach algebras
- Invertibility of operators
- The small-gain theorem
- The spectrum of an operator
- Adjoint operators
- Signal spaces $L_2$ and $H_2$
- The Fourier and Laplace transforms
- Time-invariance and causality
- Operator spaces $L_\infty$ and $H_\infty$
Operators

For Banach spaces $\mathcal{V}$ and $\mathcal{Z}$, the map $F : \mathcal{V} \to \mathcal{Z}$ is a bounded linear operator if

- **Linearity**: $F(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 F(v_1) + \alpha_2 F(v_2)$ for all $v_1, v_2 \in \mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

- **Boundedness**: There exists $K > 0$ such that $\|F(v)\| \leq K\|v\|$ for all $v \in \mathcal{V}$.

Sets of linear operators

- $\mathcal{L}(\mathcal{V}, \mathcal{Z})$ is the set of all bounded linear operators mapping $\mathcal{V}$ to $\mathcal{Z}$.

- $\mathcal{L}(\mathcal{V})$ is the set of all bounded linear operators mapping $\mathcal{V}$ to itself.

The set $\mathcal{L}(\mathcal{V}, \mathcal{Z})$ is a **Banach space**.

- It is a vector space; we have addition and scalar multiplication. e.g.
  
  $$(F_1 + F_2)(v) = F_1(v) + F_2(v)$$

- It has a norm – the induced norm.

- It is **complete**. We will not prove this here.
Banach algebras

As well as being a normed vector space, the set $\mathcal{L}(\mathcal{V})$ has additional structure, since one may compose maps. We write $(F_1F_2)(v) = F_1(F_2(v))$, giving

$$F_1, F_2 \in \mathcal{L}(\mathcal{V}) \quad \Rightarrow \quad F_1F_2 \in \mathcal{L}(\mathcal{V})$$

The space $\mathcal{L}(\mathcal{V})$ is called a *Banach algebra*

**Axiomatic definition of a Banach algebra**

- There exists an element $I \in \mathcal{B}$, such that $F \cdot I = I \cdot F = F$, for all $F \in \mathcal{B}$.
- $F(GH) = (FG)H$, for all $F, G, H$ in $\mathcal{B}$.
- $F(G + H) = FG + FH$, for all $F, G, H$ in $\mathcal{B}$.
- For all $F, G$ in $\mathcal{B}$, and each scalar $\alpha$, we have $F(\alpha G) = (\alpha F)G = \alpha FG$.
- The submultiplicative inequality: $\|FG\| \leq \|F\| \|G\|$.
The submultiplicative inequality

The submultiplicative inequality is

\[ \|FG\| \leq \|F\| \|G\| \]

- This is very useful in control; leads to a useful robustness test.
- It follows from the definition of the induced-norm:

\[ \|FGx\| \leq \|F\| \|Gx\| \leq \|F\| \|G\| \|x\| \]

Examples

- The set of linear operators on any Banach space \( \mathcal{V} \) forms a Banach algebra.
- The set of \( n \times n \) matrices forms a Banach algebra.
Invertibility of operators

An operator $F \in \mathcal{L}(\mathcal{V})$ is called invertible if there exists $G \in \mathcal{L}(\mathcal{V})$ such that

$$FG = I \text{ and } GF = I$$

We write $G = F^{-1}$ as usual.

Note that the inverse $G$ must be bounded, and that we need both equations to hold. For example, $\ell_2$ is the space of square-summable sequences

$$\ell_2 = \left\{ (x_0, x_1, \ldots) ; x_i \in \mathbb{R}^n, \sum_{i=0}^{\infty} x_i^* x_i \text{ is finite} \right\}$$

Consider the forward shift operator $Z \in \mathcal{L}(\ell_2)$ where $y = Zx$ if

$$y_k = \begin{cases} x_{k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}$$

This maps $(10, 3, 2, \ldots)$ to $(0, 10, 3, 2, \ldots)$. The backward shift operator $B \in \mathcal{L}(\ell_2)$ defined by

$$y = Bx \quad \text{if } y_k = x_{k+1} \text{ for all } k \geq 0$$

satisfies $BZ = I$, but not $ZB = I$. The operator $Z$ is called not invertible or singular, even though given $y$ one can find $x$. 
The small-gain theorem

Suppose $Q$ is an element of a Banach algebra $\mathcal{B}$. Then

$$\|Q\| < 1 \implies I - Q \text{ is invertible, and } (I - Q)^{-1} = \sum_{i=0}^{\infty} Q^i$$

Examples

- If $Q = \begin{bmatrix} 0 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$, then $\|Q\| = \sigma(Q) = 0.72$. Then we know that $I - Q$ is invertible.

- Clearly the reverse implication does not hold. For example, $Q = 2I$.

Notes

- Here we are only assuming that $Q$ is an element of a Banach algebra. We do not need to use any properties of $Q$ as a linear map.

- The submultiplicative property implies $\|PQ\| \leq \|P\| \|Q\|$. Hence if $\|P\| \leq 1$

  $$I - PQ \text{ is invertible for all operators } Q \text{ with } \|Q\| < 1.$$  

  This is very useful when analyzing stability of feedback loops.
Left and right inverses

For $F \in \mathcal{B}$, the operator $L \in \mathcal{B}$ is called the \textit{left-inverse} of $F$ if $LF = I$. Similarly, $R \in \mathcal{B}$ is called the \textit{right-inverse} of $F$ if $FR = I$.

If $F$ has both a left-inverse and a right-inverse, then they are equal, since

$$L = L(BR) = (LB)R = R$$

Series convergence

The infinite sum is defined by

$$\sum_{i=0}^{\infty} Q^i = \lim_{n \to \infty} T_n$$

where $T_n$ is the \textit{partial sum}

$$T_n = \sum_{i=0}^{n} Q^i$$
Proof of the small-gain theorem

First, we show $\sum_{i=0}^{\infty} Q^i$ is in the Banach algebra $\mathcal{B}$. We need to show that $\{T_0, T_1, \ldots\}$ is a Cauchy sequence. For $m > n$,

$$\|T_m - T_n\| = \left\| \sum_{i=n+1}^{m} Q^i \right\| \leq \sum_{i=n+1}^{m} \|Q^i\| \leq \sum_{i=n+1}^{m} \|Q\|^i$$

Recall the geometric series sum $\sum_{i=n+1}^{m} a^i = \frac{a^{n+1}(1 - a^{m-n})}{1 - a}$. Then

$$\|T_m - T_n\| \leq \frac{\|Q\|^{n+1}}{1 - \|Q\|}$$

which implies $\{T_0, T_1, \ldots\}$ is Cauchy.

Now we show that $\sum_{i=0}^{\infty} Q^i$ is the right-inverse of $I - Q$.

$$(I - Q) \sum_{k=0}^{\infty} Q^k = \sum_{k=0}^{\infty} Q^k - Q \sum_{k=0}^{\infty} Q^k$$

$$= I + \sum_{k=1}^{\infty} Q^k - Q \sum_{k=0}^{\infty} Q^k = I$$

Similarly, $\sum_{i=0}^{\infty} Q^i$ is the left-inverse of $I - Q$ also, and hence it is the inverse of $I - Q$. 
The spectrum

Suppose $F \in \mathcal{L}(\mathcal{V})$. The spectrum of $F$ is

$$\text{spec}(F) = \{ \lambda \in \mathbb{C} ; (\lambda I - F) \text{ is not invertible} \}$$

The spectral radius of $F$ is

$$\rho(F) = \max\{|\lambda| ; \lambda \in \text{spec}(F)\}.$$

We say $\lambda$ is an eigenvalue of $F$ if there exists $x \in \mathcal{V}$ such that

$$Fx = \lambda x$$

Clearly, if $\lambda$ is an eigenvalue of $F$, then $\lambda \in \text{spec}(F)$. But the converse is not true in general. In general

$$\{ \lambda \in \mathbb{C} ; \lambda \text{ is an eigenvalue of } F \} \subseteq \text{spec}(F)$$

These sets are equal for finite-dimensional matrices.
The spectral radius and the norm

The spectral radius satisfies

$$\rho(F) \leq \|F\|$$

for all operators $F \in \mathcal{L}(\mathcal{V})$.

Proof

For matrices, one can see this by considering an eigenvector. But in general $F$ may not have eigenvectors.

Suppose $|\lambda| > \|F\|$. Then, set $Q = \lambda^{-1}F$, and then $\|Q\| < 1$, which implies that $I - Q$ is invertible by the small-gain theorem.

Also, if $I - Q$ is invertible, so is $\lambda(I - Q)$, which is

$$\lambda(I - Q) = \lambda(I - \lambda^{-1}F) = \lambda I - F$$

Hence $\lambda \not\in \text{spec}(F)$. 

The spectrum of a product

Consider operators $P \in \mathcal{L}(U, V)$ and $Q \in \mathcal{L}(V, U)$. Then

$$(I - PQ) \text{ is invertible } \iff (I - QP) \text{ is invertible}$$

**Proof:** If $I - PQ$ is invertible, we can construct the inverse of $I - QP$ according to

$$(I - QP)^{-1} = I + Q(I - PQ)^{-1}P$$

This is called the *Sherman-Morrison-Woodbury* formula, or the *Matrix-inversion lemma*. It can be shown directly by multiplying both sides by $I - QP$.

The spectrum of a product

An immediate consequence is that, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$,

$$\lambda \in \text{spec}(PQ) \iff \lambda \in \text{spec}(QP)$$

**Proof:**

$$\lambda I - PQ \text{ is invertible } \iff I - \lambda^{-1}PQ \text{ is invertible}$$

$$\iff I - \lambda^{-1}QP \text{ is invertible}$$

$$\iff \lambda I - QP \text{ is invertible}$$
The spectrum of a product

Example

\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \implies \quad PQ = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}
\]

The adjoint operator

Suppose \( \mathcal{V} \) and \( \mathcal{Z} \) are Hilbert spaces, and \( F \in \mathcal{L}(\mathcal{V}, \mathcal{Z}) \). The operator \( F^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V}) \) is called the adjoint of \( F \) if

\[
\langle z, Fv \rangle = \langle F^*z, v \rangle
\]

for all \( v \in \mathcal{V} \) and \( z \in \mathcal{Z} \).

Properties

- \( \|F^*\| = \|F\| = \|F^*F\|^{\frac{1}{2}} \)
- \( \|F\|^2 = \rho(F^*F) \).

Example

The adjoint of a matrix is the complex conjugate transpose.
Self-adjoint operators

The operator $F$ is called \textit{self-adjoint} or \textit{hermitian} if $F = F^*$. 

- If $F$ is self-adjoint, then $\rho(F) = \|F\|$.
- The \textit{quadratic form} $\langle Fv, v \rangle$ takes only real values.
- If $\lambda \in \text{spec}(F)$, then $\lambda \in \mathbb{R}$.

Positive operators

A self-adjoint operator $F$ is called \textit{positive semidefinite}, written $F \geq 0$, if

$$\langle Fv, v \rangle \geq 0 \quad \text{for all } v$$

A self-adjoint operator $F$ is called \textit{positive definite}, written $F > 0$, if

there exists $\varepsilon > 0$ such that $\langle Fv, v \rangle \geq \varepsilon \|v\|^2$ for all $v$

For matrices, this coincides with the usual definition of positive definiteness. If $F \in \mathbb{R}^{n \times n}$

$$F > 0 \quad \implies \quad \langle Fv, v \rangle = v^* Fv \geq \frac{\lambda_{\min}(F)}{2} v^*v$$
Isometric operators

The operator $U$ is called isometric if $U^*U = I$.

Properties

- Angles are preserved: $\langle Uv_1, Uv_2 \rangle = \langle U^*Uv_1, v_2 \rangle = \langle v_1, v_2 \rangle$
- Norms are preserved: $\|Uv\| = \|v\|$ for all $v$.
- Distances are preserved: $\|Uv_1 - Uv_2\| = \|v_1 - v_2\|$.

Unitary operators

The operator $U$ is called unitary if

$$U^*U = I \quad \text{and} \quad UU^* = I$$

A unitary operator $U : U \to V$ is called an isomorphism.

Example

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is isometric, but not unitary
The \( L_2 \) spaces

The Hilbert space \( L_2(-\infty, \infty) \) is the set of functions \( u : \mathbb{R} \rightarrow \mathbb{C}^n \) with inner product

\[
\langle u, v \rangle = \int_{-\infty}^{\infty} u(t)^* v(t) \, dt
\]

The Hilbert space \( \hat{L}_2(j\mathbb{R}) \) is the set of functions \( \hat{u} : j\mathbb{R} \rightarrow \mathbb{C}^n \) with inner product

\[
\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{v}(j\omega) \, dt
\]

The Fourier Transform

The Fourier transform is a map \( \Phi : L_2(-\infty, \infty) \rightarrow \hat{L}_2(j\mathbb{R}) \) defined by

\[
\Phi : u \mapsto \hat{u} \quad \hat{u}(j\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} \, dt
\]

- \( \Phi \) is a bounded linear operator.
- \( \Phi \) is invertible. The inverse is given by \( u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega) e^{j\omega t} \, d\omega \).
- \( \Phi \) is unitary. It is an isomorphism between \( L_2(-\infty, \infty) \) and \( \hat{L}_2(j\mathbb{R}) \).
Frequency domain spaces

The open right-half plane and closed right half-plane are

\[ \mathbb{C}^+ = \left\{ z \in \mathbb{C} ; \text{Re}(z) > 0 \right\} \quad \text{and} \quad \overline{\mathbb{C}}^+ = \left\{ z \in \mathbb{C} ; \text{Re}(z) \geq 0 \right\} \]

The space \( H_2 \)

The space \( H_2 \) is the set of functions \( \hat{u} : \overline{\mathbb{C}}^+ \to \mathbb{C}^n \) for which

- \( \hat{u} \) is analytic in the open right-half plane \( \mathbb{C}^+ \).
- For almost every real number \( \omega \),
  \[ \hat{u}(j\omega) = \lim_{\sigma \to 0^+} \hat{u}(\sigma + j\omega) \]
- The maximum integral over a vertical line \( \text{Re}(z) = \sigma \) in \( \overline{\mathbb{C}}^+ \)
  \[ \sup_{\sigma \geq 0} \int_{-\infty}^{\infty} \left\| \hat{u}(\sigma + j\omega) \right\|^2_2 d\omega \quad \text{is finite} \]

Rational functions

A rational function \( \hat{u} \) is in \( H_2 \) if it is strictly proper and has no poles in the closed right-half plane.
The Laplace transform

The Laplace transform $\Lambda : u \mapsto \hat{u}$ is defined by

$$\hat{u}(s) = \int_0^\infty u(t)e^{-st}dt$$

Notes

- $\Lambda : L_2[0, \infty) \rightarrow H_2$
- $\Lambda$ is a bounded linear operator.
- $\Lambda$ is invertible. It is an isometric isomorphism between $L_2[0, \infty)$ and $H_2$.

The inner product in $H_2$

Given a function $\hat{u} \in H_2$, this defines a function on the imaginary axis which is an element of $\hat{L}_2(j\mathbb{R})$. We define the inner product of two functions in $H_2$ to be their inner product as elements of $\hat{L}_2(j\mathbb{R})$. That is

$$\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{u}^*(j\omega)\hat{v}(j\omega)\,d\omega$$

Note that $H_2$ is a subspace of $L_2(j\mathbb{R})$. 
Summary of signal spaces

The signal spaces are

\[ L_2(-\infty, \infty) \supset L_2[0, \infty) \]

\[ \hat{L}_2(j\mathbb{R}) \supset H_2 \]

\[ \Phi \quad \Phi^{-1} \]

\[ \Lambda \quad \Lambda^{-1} \]
The space $\hat{L}_\infty(j\mathbb{R})$

Consider the set of matrix-valued functions

$$\hat{L}_\infty = \left\{ G : j\mathbb{R} \rightarrow \mathbb{C}^{p \times m} ; \|G\|_\infty = \text{ess sup}_{\omega \in \mathbb{R}} \sigma(\hat{G}(j\omega)) \text{ is finite} \right\}$$

We will use this as a space of transfer functions.

Multiplication operators

The function $G \in L_\infty(j\mathbb{R})$ defines a multiplication operator $M_{\hat{G}} : L_2(j\mathbb{R}) \rightarrow L_2(j\mathbb{R})$

$$\hat{y} = M_{\hat{G}} \hat{u} \quad \iff \quad \hat{y}(j\omega) = \hat{G}(j\omega) \hat{u}(j\omega)$$

Notes

- $\hat{G}$ is our usual notion of transfer function
- Using the Fourier transform, $\hat{G}$ defines a map $G : L_2(-\infty, \infty) \rightarrow L_2(-\infty, \infty)$ by
  $$G = \Phi^{-1} M_{\hat{G}} \Phi$$
- If $\hat{G}$ is rational, then $\hat{G} \in \hat{L}_\infty(j\mathbb{R})$ if and only if it is proper and has no poles on the imaginary axis.
The shift operator

The shift operator $S_\tau : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined by

$$y = S_\tau u \iff y(t) = u(t - \tau)$$

This is also called the $\tau$-delay.

Time-invariance

An operator $G : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is called time-invariant if

$$GS_\tau = S_\tau G \quad \text{for all } \tau \geq 0$$

Theorem

An operator $G : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is time-invariant if and only there exists a function $\hat{G} \in \hat{L}_\infty(j\mathbb{R})$ such that the multiplication operator satisfies

$$G = \Phi^{-1}M_{\hat{G}}\Phi$$
The truncation operator

The truncation operator $P_\tau : L_2(-\infty, \infty) \rightarrow L_2(-\infty, \infty)$ is defined by

$$ y = P_\tau u \iff y(t) = \begin{cases} u(t) & \text{for } t \leq \tau \\ 0 & \text{otherwise} \end{cases} $$

Causality

The operator $G : L_2(-\infty, \infty) \rightarrow L_2(-\infty, \infty)$ is called causal if

$$ P_\tau GP_\tau = P_\tau G \quad \text{for all } \tau \in \mathbb{R} $$

Interpretation

$y_1 = P_\tau Gu$ is the output signal on $(-\infty, \tau)$ corresponding to input $u$.

$y_2 = P_\tau GP_\tau u$ is the output signal on $(-\infty, \tau)$ corresponding to input $P_\tau u$.

If $y_1 = y_2$, then the output before time $\tau$ is unaffected by inputs after time $\tau$. 
Time-invariance and causality

If \( G \) is time-invariant, then it is causal if and only if
\[
P_0GP_0 = P_0G
\]
That is, we need only check the causality condition \( P_\tau GP_\tau = P_\tau G \) at \( \tau = 0 \).

Corollary

A time-invariant operator \( G \) is causal if and only if
\[
u \in L_2[0, \infty) \implies Gu \in L_2[0, \infty)
\]

Notes

• This follows from \( P_0G(I - P_0) = 0 \).
The space $H_\infty$

The set of matrix-valued functions $G' : \bar{\mathbb{C}}^+ \rightarrow \mathbb{C}^{p \times m}$ satisfying the following properties:

- $\hat{G}(s)$ is analytic in $\mathbb{C}^+$;
- For almost every real number $\omega$
  \[ \lim_{\sigma \to 0^+} \hat{G}(\sigma + j\omega) = \hat{G}(j\omega) \]
- $\sup_{s \in \bar{\mathbb{C}}^+} \sigma(\hat{G}(s))$ is finite.

Notes

- The norm on $H_\infty$ is given by $\|G\|_\infty = \text{ess sup}_{\omega \in \mathbb{R}} \sigma(\hat{G}(j\omega))$
- If $\hat{G}$ is rational, then $\hat{G} \in H_\infty$ if and only if it is proper and has no poles in the closed right-half of the complex plane.
**Theorem**

- Every \( \hat{G} \in H_\infty \) defines a causal, time-invariant operator \( G : L_2[0, \infty) \rightarrow L_2[0, \infty) \). 
  \[ z = Gu \]  
  is defined by the multiplication operator  
  \[ \hat{z}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega) \]

- If \( G : L_2[0, \infty) \rightarrow L_2[0, \infty) \) is bounded, linear, and time-invariant, then there exists a function \( \hat{G} \in H_\infty \) such that  
  \[ z = Gu \iff \hat{z}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega) \]

**Notes**

- There is a one-to-one correspondence between functions in \( H_\infty \) and linear time-invariant (LTI) systems.

- We denote the subset of rational functions in \( H_\infty \) by \( RH_\infty \).

- Every function \( \hat{G} \in RH_\infty \) can be expressed as  
  \[ \hat{G}(s) = C(sI - A)^{-1}B + D \]
  for some matrices \( A, B, C, D \).