Engr210a Lecture 7: System models and model reduction

- Correspondence between state-space systems and transfer functions
- Stability and minimal realizations
- The induced norm
- The H_∞ norm
- Bode plots
- Measuring the difference between systems
- Additive uncertainty
- Model reduction

State-space systems

Suppose (A, B, C, D) is a stable state-space system. Construct the transfer function

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

- The transfer function $\hat{G} \in H_\infty$, since it is analytic and bounded in $\bar{C}^+$ and continuous along the imaginary axis.
- Hence the multiplication operator mapping $M_{\hat{G}} : H_2 \to H_2$ defined by

  $$\hat{y} = M_{\hat{G}}\hat{u} \qquad \Longleftrightarrow \qquad \hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$

  is a bounded linear operator on H_2.
- The system is causal, and time-invariant because multiplication operators defined by elements of H_∞ define causal and time-invariant linear systems.
- H_2 is isomorphic to $L_2[0, \infty)$ via the Laplace transform $\Lambda : L_2[0, \infty) \to H_2$, so the operator G defined by

  $$G = \Lambda^{-1} M_{\hat{G}} \Lambda$$

  is a *bounded linear operator* on $L_2[0, \infty)$.

Conclusion: Every stable state-space linear system defines a bounded linear operator on the space of signals $L_2[0, \infty)$.

State-space systems

Suppose the map $G : L_2[0, \infty) \to L_2[0, \infty)$ is bounded, linear, and time-invariant.

- G defines a bounded linear operator $\check{G} : H_2 \to H_2$ via the Laplace transform
$$\check{G} = \Lambda G \Lambda^{-1}$$
- Since G is linear and time-invariant, $\check{G}$ is the multiplication operator corresponding to a function $\hat{G} \in H_\infty$.
$$\check{G} = M_{\hat{G}} \qquad \hat{y} = M_{\hat{G}} \hat{u} \quad \Longleftrightarrow \quad \hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$
- If the function $\hat{G}$ is rational, then it has a minimal state-space realization (A, B, C, D) which satisfies
$$\hat{G} = C(sI - A)^{-1}B + D$$
- Since $\hat{G} \in H_\infty$, the function $\overline{\sigma}(\hat{G}(\cdot))$ is bounded in the closed right-half plane. This implies that $\hat{G}$ has no poles in the closed right-half plane.
- This implies that the system
$$\dot{x}(t) = Ax(t)$$
is stable, which we show next.

Stability

If (A, B, C, D) is a minimal realization for a transfer function $\hat{G}(s)$, and $\hat{G}$ has no poles in the closed right-half plane, then the system

$$\dot{x}(t) = Ax(t)$$

is stable.

Recall facts

- We say $\hat{G} : \mathbb{C} \to \mathbb{C}^{p \times m}$ has a pole at $\lambda \in \mathbb{C}$ if there is some i, j so that the element

  $$\lim_{s \to \lambda} |\hat{G}_{ij}(s)| = \infty$$

  This is equivalent to

  $$\lim_{s \to \lambda} \overline{\sigma}(G)(s) = \infty$$

- The system

  $$\dot{x}(t) = Ax(t)$$

  is stable if and only if all eigenvalues of A have strictly negative real part; that is

  $$\lambda \in \operatorname{spec}(A) \qquad \Longrightarrow \qquad \operatorname{Re}(\lambda) < 0$$

  In this case the matrix A is called a *Hurwitz* matrix.

Simple case

Suppose A has only one eigenvalue λ_1, possibly repeated. Then if (A, B, C, D) is a minimal realization for $\hat{G}$, then λ_1 is a pole of $\hat{G}$.

Proof

$\hat{G}$ is a proper rational function, and if λ is a pole of G then λ is an eigenvalue of A. Hence either there is an element of $\hat{G}$ such that

$$\hat{G}_{ij}(s) = \frac{c_1 s + c_0}{s - \lambda_1}$$

with $c_1\lambda_1 + c_0 \neq 0$, or $\hat{G}$ is just a constant matrix, say $\hat{G}(s) = G_0$. But if that were the case, then we would be able to realize $\hat{G}$ with the realization $(\emptyset, \emptyset, \emptyset, G_0)$, a zero'th order realization, contradicting the assumption that (A, B, C, D) is minimal.

General case

Suppose (A, B, C, D) is a minimal realization for $\hat{G}$. Then if $\lambda \in \operatorname{spec}(A)$, then λ is a pole of $\hat{G}$.

Proof: Choose coordinates so that A is in Jordan form

$$A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{bmatrix}$$

Choose the blocks so that each J_i has only one eigenvalue, λ_i, and $\lambda_i \neq \lambda_j$ if $i = j$. Partition B and C compatibly with A so that

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & \dots & C_q \end{bmatrix}$$

Then

$$C(sI - A)^{-1}B + D = \sum_{i=1}^{q} \Big(C_i(sI - J_i)^{-1} B_i \Big) + D$$

By our previous argument, $C_i(sI - J_i)^{-1}B_i$ must have a pole at λ_i, and $\lambda_i \neq \lambda_j$ so terms in different blocks cannot cancel.

State-space systems

We can think of systems in three ways

- **Bounded linear operators**

    For every causal time-invariant bounded linear operator on $L_2[0,\infty)$ there is a corresponding function in H_∞.

- **Functions in** H_∞.

    For every *rational* function in H_∞, there is a corresponding stable state-space system.

    (There are also some unstable ones, whose unstable states are uncontrollable or unobservable, but any minimal realization will be stable.)

- **State-space realizations**

    For every stable, linear time-invariant state-space system there is a causal time-invariant bounded linear operator on $L_2[0,\infty)$.

    The corresponding H_∞ function is rational.

Norms on systems

The abbreviation LTI stands for *linear, time-invariant*.

We now have two norms on stable LTI systems $G : L_2[0, \infty) \to L_2[0, \infty)$.

- Since H_∞ is a Banach space, we have the norm

$$\|G\|_\infty = \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \overline{\sigma}(\hat{G}(j\omega))$$

- The induced norm on $L_2[0, \infty)$

$$\|G\| = \sup_{\substack{u \in L_2[0,\infty) \\ u \neq 0}} \frac{\|Gu\|}{\|u\|}$$

Theorem

These two norms are equal.

Theorem

The H_∞ norm is equal to the induced $L_2[0,\infty)$ norm.

$$\sup_{\omega\in\mathbb{R}} \|\hat{G}(j\omega)\| = \sup_{\omega\in\mathbb{R}} \overline{\sigma}(\hat{G}(j\omega)) = \|\hat{G}\|_\infty = \|G\| = \sup_{\substack{u\in L_2[0,\infty)\\ u\neq 0}} \frac{\|Gu\|}{\|u\|}$$

Proof: First, we prove $\|G\| \leq \|\hat{G}\|_\infty$. Suppose $y = Gu$. Then, taking Laplace transforms, $\hat{y}, \hat{u} \in H_2$, and $\hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$.

Since the Laplace transform is isometric,

$$\begin{aligned}
\|y\|^2 = \|\hat{y}\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{y}(j\omega)\|^2 \, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{G}(j\omega)\hat{u}(j\omega)\|^2 \, d\omega \\
&\leq \frac{1}{2\pi}\|\hat{G}(j\omega)\|_\infty^2 \int_{-\infty}^{\infty} \|\hat{u}(j\omega)\|^2 \, d\omega \\
&= \|\hat{G}(j\omega)\|_\infty^2 \|\hat{u}\|^2 \\
&= \|\hat{G}(j\omega)\|_\infty^2 \|u\|^2
\end{aligned}$$

Proof continued

Now we prove that $\|G\| \geq \|\hat{G}\|_\infty$.

Given $\varepsilon > 0$, we need to construct a signal $u \in L_2[0,\infty)$ such that

$$\|y\|_2 \geq (\|\hat{G}\|_\infty - \varepsilon)\|u\|_2$$

Since $\hat{G} \in H_\infty$ and $H_\infty \subset L_\infty$, we have $\hat{G} \in L_\infty$. Then $\hat{G}$ defines a causal LTI operator on $L_2(-\infty,\infty)$. Taking Fourier transforms, this is defined by multiplication

$$\hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$

where $\hat{y}, \hat{u} \in L_2(j\mathbb{R})$.

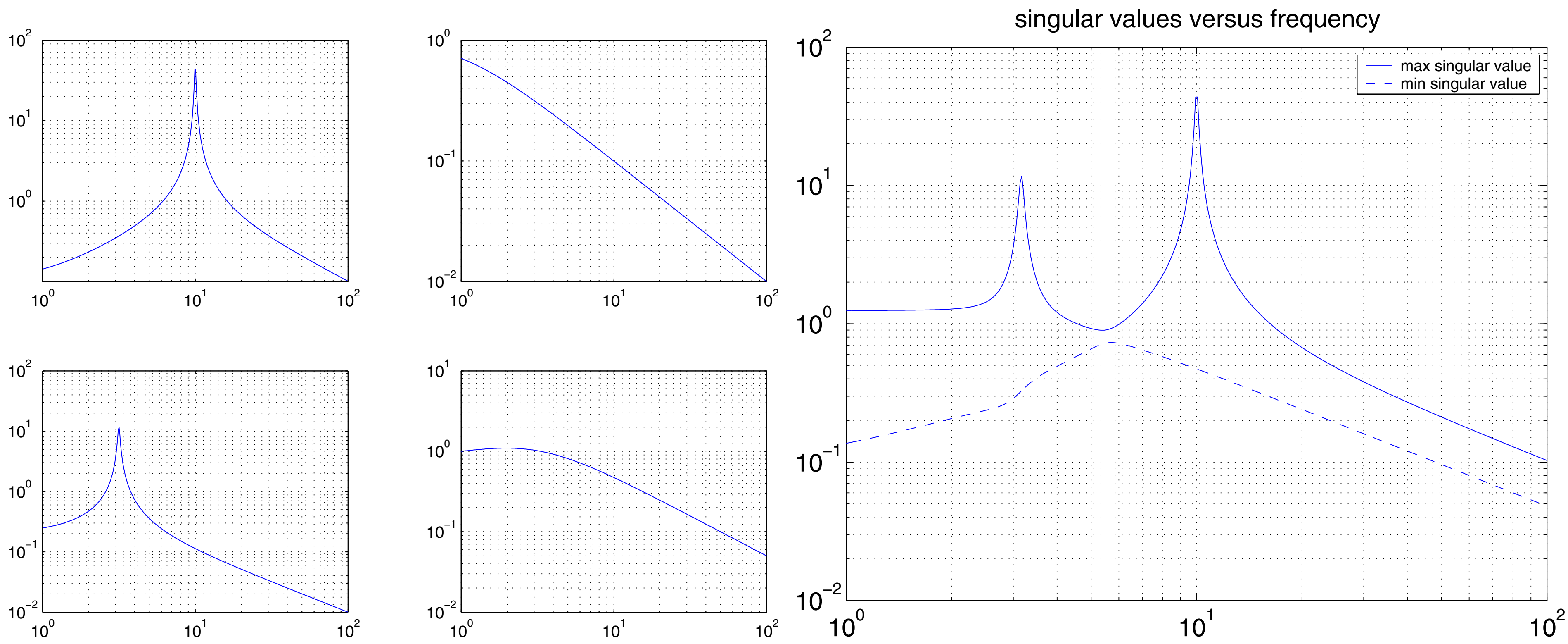
Choose a function $\hat{u}$ which has a narrow peak such that

$$\|\hat{y}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{G}(j\omega)\hat{u}(j\omega)\|^2\, d\omega \geq (\|\hat{G}\|_\infty - \varepsilon)^2\|\hat{u}\|^2$$

Now $\hat{u} = \Phi u$, the Fourier transform of $u \in L_2(-\infty,\infty)$. Therefore $u(t) \to 0$ as $t \to \infty$, and we can truncate it at a sufficiently negative time $\tau \ll 0$ and it will still satisfy the above inequality. Set u_2 equal to this truncation, $u_2 = (I - P_\tau)u$, and let $u_3 = S_\tau u_2$, which is the same signal shifted forward so that $u_3 \in L_2[0,\infty)$. Then u_3 also satisfies the inequality, and $Gu_3 \in L_2[0,\infty)$.

Bode Plots

$$\hat{G}(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2+0.2s+100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2+0.1s+10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix} \qquad \|G\| = 50.25$$



The induced-norm

- $\|G\|$ is called the *induced-norm* or the *H-infinity* norm of G.
- If G is stable, then $\hat{G} \in H_\infty$, so $\|G\|$ is finite, and

$$\|\hat{G}\|_\infty = \sup_{s \in \bar{\mathbb{C}}^+} \overline{\sigma}(\hat{G}(s)) = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(\hat{G}(j\omega))$$

- If G is unstable, then the induced-norm $\|G\|$ is not finite, and

$$\sup_{s \in \bar{\mathbb{C}}^+} \overline{\sigma}(\hat{G}(s))$$

  is not finite.

Caveat

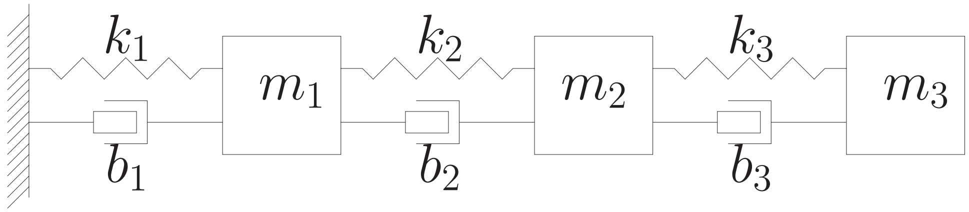
If G is unstable, then

$$\sup_{\omega \in \mathbb{R}} \overline{\sigma}(\hat{G}(j\omega)) = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(C(j\omega I - A)^{-1}B + D)$$

may be finite. Even if $\hat{G}$ is not analytic in the closed right-half plane and hence $\hat{G} \notin H_\infty$, we can still have $G \in L_\infty(j\mathbb{R})$.

The induced-norm

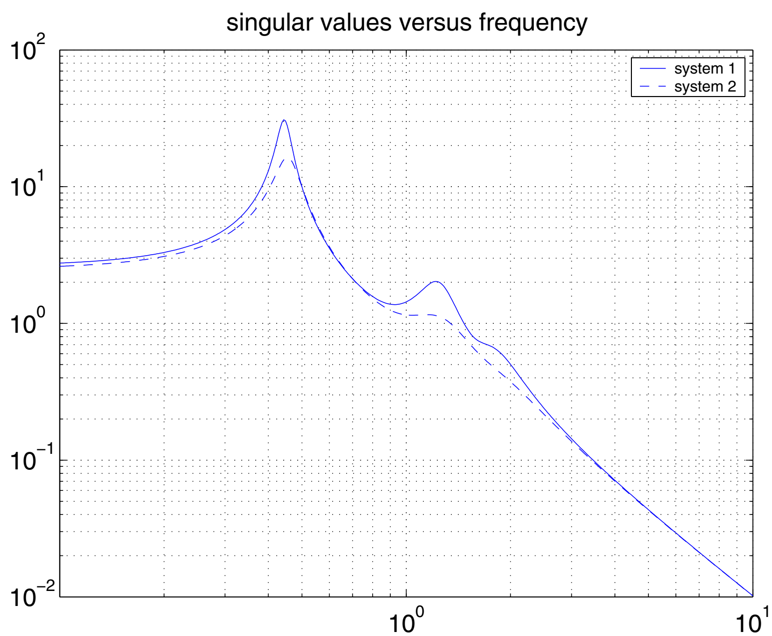
An important use of the norm is in measuring the difference between two systems.

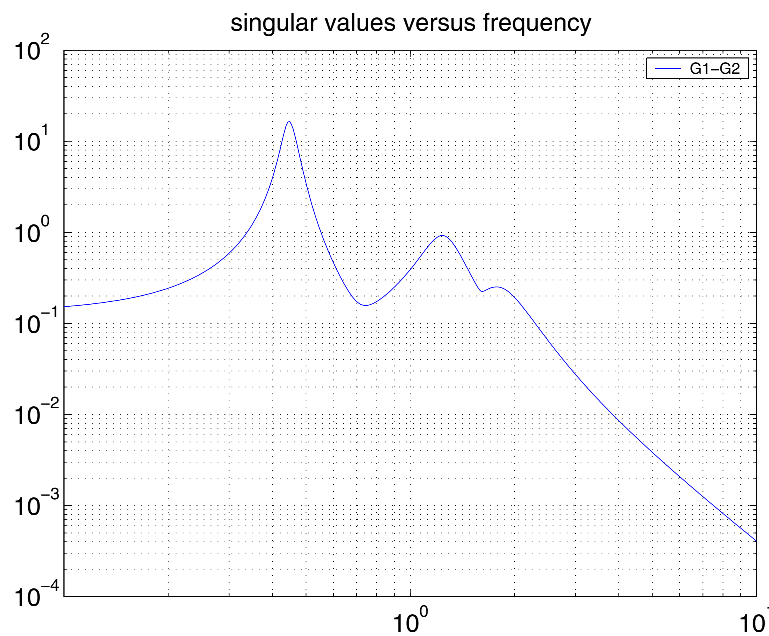


Example: 2 inputs, 2 output system. Inputs are forces applied to masses 1 and 3, outputs are positions of masses 1 and 2.

G_1 has $m_i = 1$, $k_i = 1$, $b_i = 0.2$. G_2 has $m_i = 0.95$, $k_i = 1$, $b_i = 0.35$.

$\|G_1\| = 30.93$, $\|G_2\| = 16.37$, $\|G_1 - G_2\| = 16.42$.





Robust control, first approach

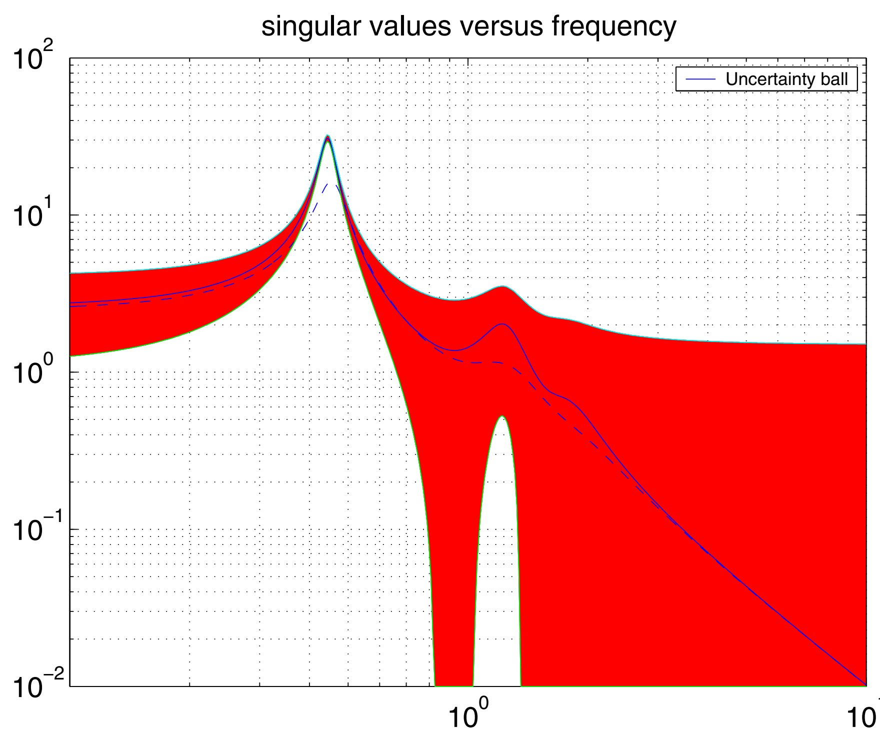
Instead of trying to design a control system for G_1 or G_2, try to design a controller that achieves a specified level of performance for any G such that

$$\|G - G_{\text{nominal}}\| < c$$

In other words, design a controller that will work for any G such that

$$G = G_{\text{nominal}} + \Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

This sounds reasonable, but leads to large uncertainty at small values of $\hat{G}(j\omega)$.

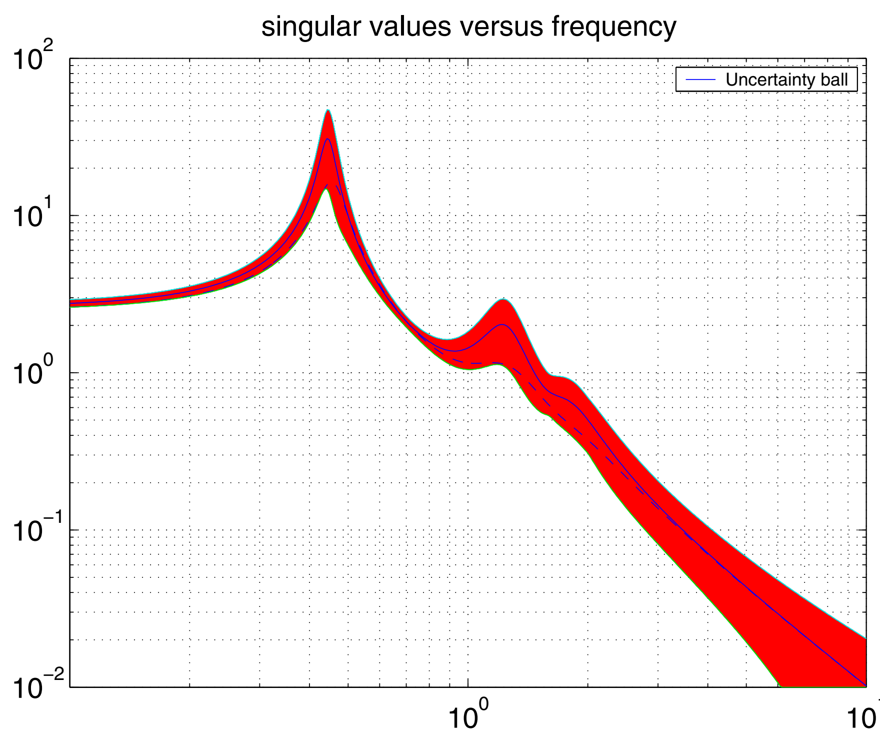


Weighted additive uncertainty

Design a controller that achieves a specified level of performance for any G such that

$$G = G_{\text{nominal}} + W\Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

Here W is a transfer function, chosen to be small at frequencies where the model is good, and large elsewhere.

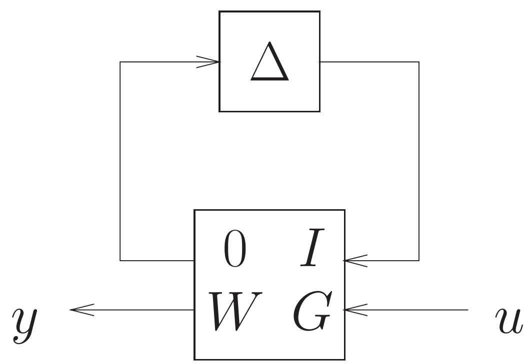


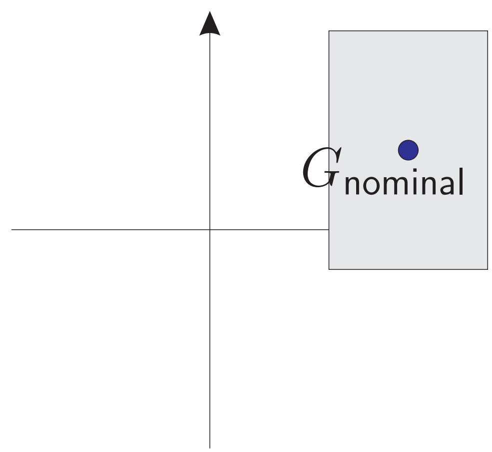
Weighted additive uncertainty

Design a controller that achieves a specified level of performance for any G such that

$$G = G_{\text{nominal}} + W\Delta \qquad \text{for some } \Delta \text{ with } \|\Delta\| < c$$

We are therefore trying to do a control design for a *set of systems*, not just a single system. This particular set is a *ball* in H_∞. It is called a *weighted additive uncertainty ball*.





We can also represent this as the above block-diagram, called a *linear-fractional transformation*.

Here the system $G = \begin{bmatrix} 0 & I \\ W & G \end{bmatrix}$ is called *the generalized plant*.

Model reduction

Suppose $G \in H_\infty$ has a minimal realization of dimension n. Given $r < n$, we would like to find the $G_{\text{reduced}} \in H_\infty$ which minimizes

$$\|G - G_{\text{reduced}}\|$$

Notes

- This problem has a long history. It is known as the *optimal H_∞ model reduction problem*.
- Since $G \in H_\infty$, this only makes sense for stable systems.
- Once we have G_{reduced}, we can use it for control design. In particular, we can design a controller robust to the error between G and G_{reduced}. Typically this requires much less computational time than designing for G.