2. Convexity and Duality

- Formulation of optimization problems
- Engineering examples
- Convex sets and functions
- Convex optimization problems
- Standard problems: LP and SDP
- Feasibility problems
- Algorithms
- Certificates and separating hyperplanes
- Duality and geometry
- Examples: LP and and SDP
- Theorems of alternatives

Optimization Problems

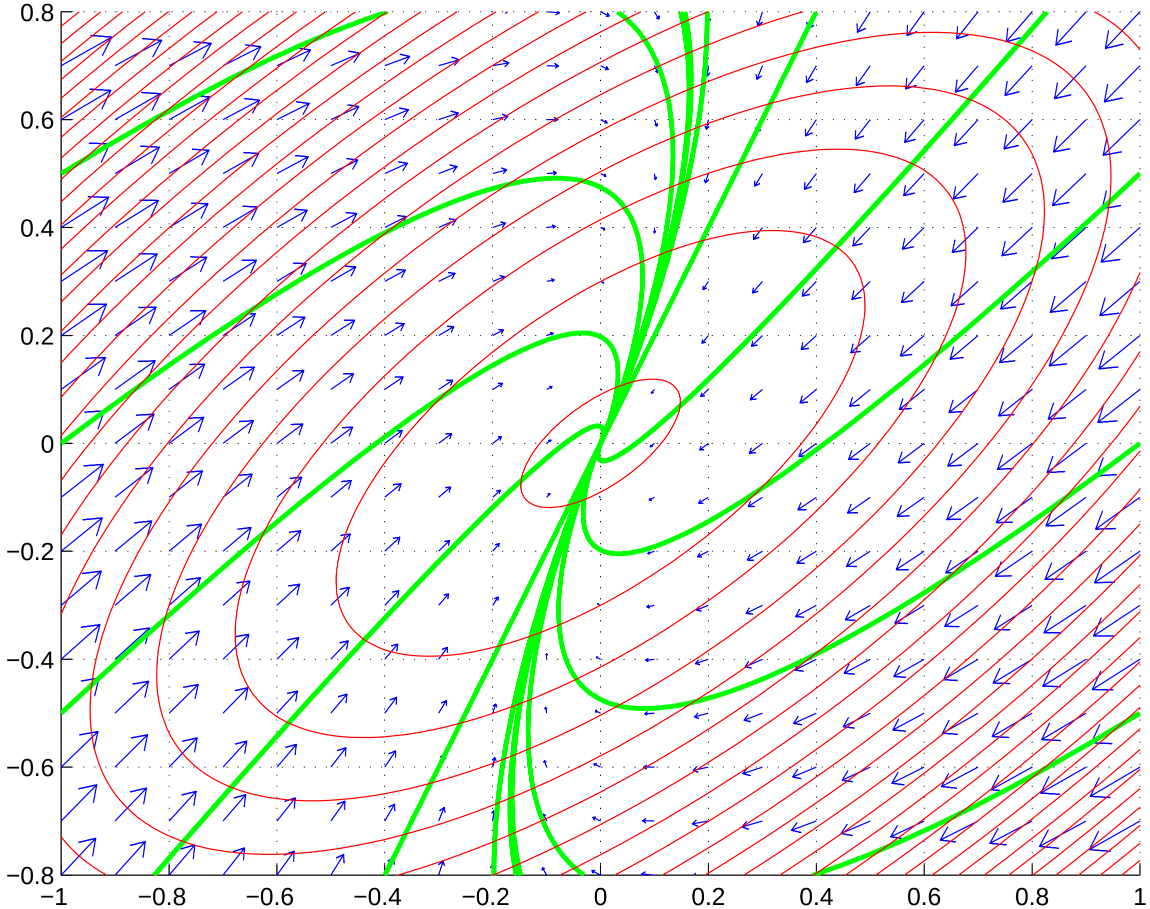
A familiar problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \dots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \dots, p \end{array}$$

- $x \in \mathbb{R}^n$ is the variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the *objective function*
- $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ define *inequality constraints*
- $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, p$ define *equality constraints*

Lyapunov functions

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -0.7 & 0.2 \\ -0.6 & -0.1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad V(x) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1.58 & -1.28 \\ -1.28 & 2.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



- Ellipsoids are the *sublevel sets* of V

$$E = \left\{ x \in \mathbb{R}^2 \mid V(x) < c \right\}$$

  These are *invariant sets* of the system.

- $V(x(t))$ is a *decreasing function* along every trajectory.

nonlinear systems

Suppose we have the system of ordinary differential equations

$$\dot{x}(t) = f(x)$$

with $x(t) \in \mathbb{R}^n$ and $f(0) = 0$. (The origin is an equilibrium point.)

global asymptotic stability

If there is a function $V : \mathbb{R}^n \to \mathbb{R}$ (called a *Lyapunov function*) which satisfies

(i) V is differentiable, and its gradient is continuous.

(ii) $V(x) > 0$ for $x \neq 0$, and $V(0) = 0$.

(iii) $\displaystyle \frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0$ for $x \neq 0$.

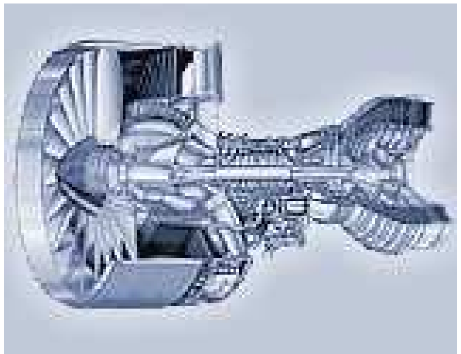
(iv) If $\{x_0, x_1, \dots\}$ is a sequence such that $\|x_k\| \to \infty$, then $V(x_k) \to \infty$.

then for any initial condition $x(0)$

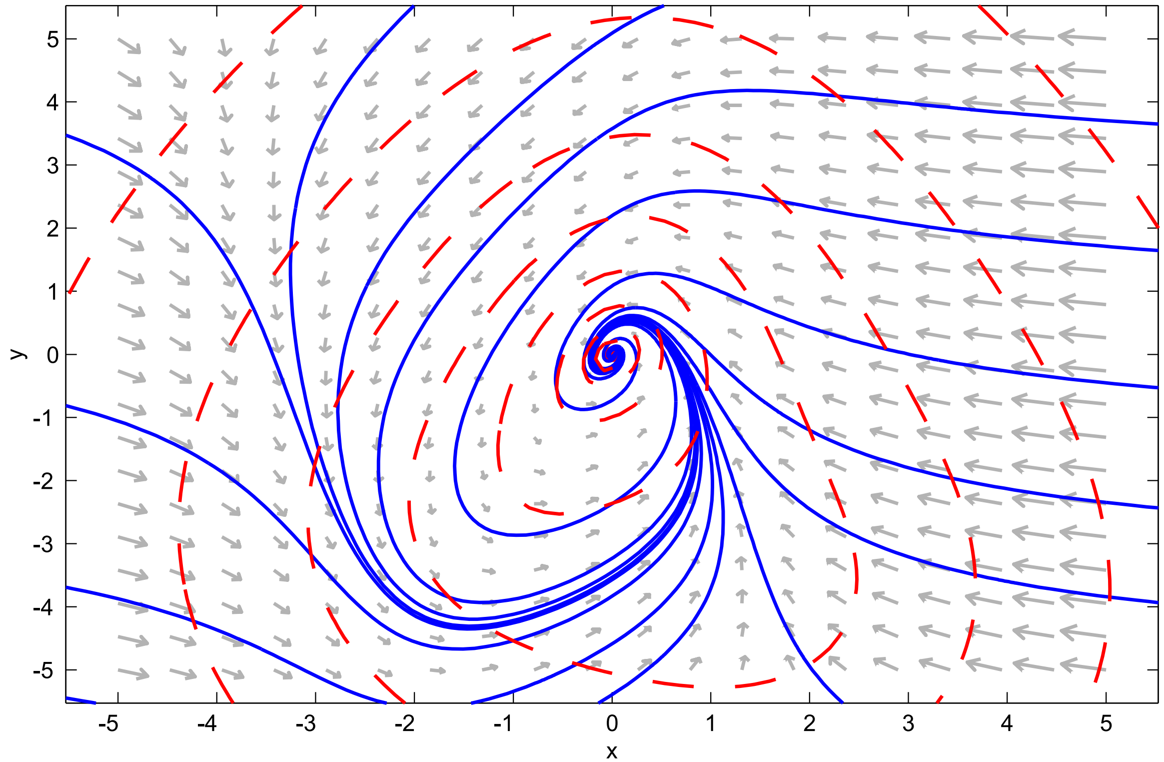
$$\boxed{\lim_{t \to \infty} x(t) = 0}$$

example: Lyapunov functions for a nonlinear system

A model for a jet engine (from Moore-Greitzer), with controller



$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$



$$V = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 + 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.000018868xy^3 + 0.090723y^4$$

computation of lyapunov functions

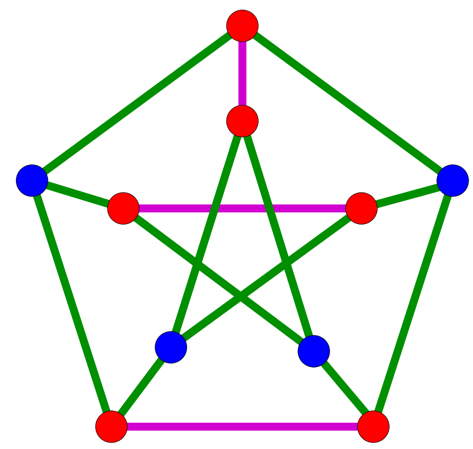
- the above Lyapunov function was found automatically
- many extensions possible
  - switching or hybrid systems
  - finite-state automata; computation of *invariants*
  - uncertain systems
  - bounds on cost via solution of Hamilton-Jacobi inequalities

Graph problems

Graph problems appear in many areas: MAX-CUT, independent set, cliques, etc.

MAX CUT partitioning

- Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets.
- Practical applications (circuit layout, etc.)
- NP-complete.



How to compute bounds, or exact solutions, for this kind of problems?

discrete problems: LQR with binary inputs

- linear discrete-time system $x(t+1) = Ax(t) + Bu(t)$ on interval $t = 0, \dots, N$
- objective is to minimize the quadratic tracking error

$$\sum_{t=0}^{N-1} \bigl(x(t) - r(t)\bigr)^T Q \bigl(x(t) - r(t)\bigr)$$

- using binary inputs

$$u_i(t) \in \{-1, 1\} \qquad \text{for all } i = 1, \dots, m, \text{ and } t = 0, \dots, N-1$$

Basic Nomenclature

A set $S \subset \mathbb{R}^n$ is called

- *affine* if $x, y \in S$ implies $\theta x + (1-\theta)y \in S$ for all $\theta \in \mathbb{R}$; i.e., the line through x, y is contained in S
- *convex* if $x, y \in S$ implies $\theta x + (1-\theta)y \in S$ for all $\theta \in [0,1]$; i.e., the line segment between x and y is contained in S.
- *a convex cone* if $x, y \in S$ implies $\lambda x + \mu y \in S$ for all $\lambda, \mu \geq 0$; i.e., the *pie slice* between x and y is contained in S.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called

- *affine* if $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$ for all $\theta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$; i.e., f is equals a linear function plus a constant $f = Ax + b$
- *convex* if $f\big(\theta x + (1-\theta)y\big) \leq \theta f(x) + (1-\theta)f(y)$ for all $\theta \in [0,1]$ and $x, y \in \mathbb{R}^n$

Examples of Convex Functions

- $f(x) = c$
- $f(x) = \lambda^T x + c$
- if $Q \succeq 0$ then $f(x) = x^T Q x$ is convex
- on $\mathbb{R}$, the exponential $f(x) = e^x$ is convex
- any norm $f(x) = \|x\|$ is convex

Properties of Convex Functions

- $f_1 + f_2$ is convex if f_1 and f_2 are
- $f(x) = \max\{f_1(x), f_2(x)\}$ is convex if f_1 and f_2 are
- $g(x) = \sup_y f(x, y)$ is convex if $f(x, y)$ is convex in x for each y
- convex functions are continuous on the interior of their domain
- $f(Ax + b)$ is convex if f is
- $Af(x) + b$ is convex if f is
- $g(x) = \inf_y f(x, y)$ is convex if $f(x, y)$ is jointly convex
- the $\alpha-$*sublevel set*

$$\{\, x \in \mathbb{R}^n \mid f(x) \leq \alpha \,\}$$

  is convex if f is convex; (the converse is not true)

Convex Optimization Problems

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \dots, m \\
& h_i(x) = 0 & \text{for all } i = 1, \dots, p
\end{array}
$$

This problem is called a *convex program* if

- the objective function f_0 is convex
- the inequality constraints f_i are convex
- the equality constraints h_i are affine

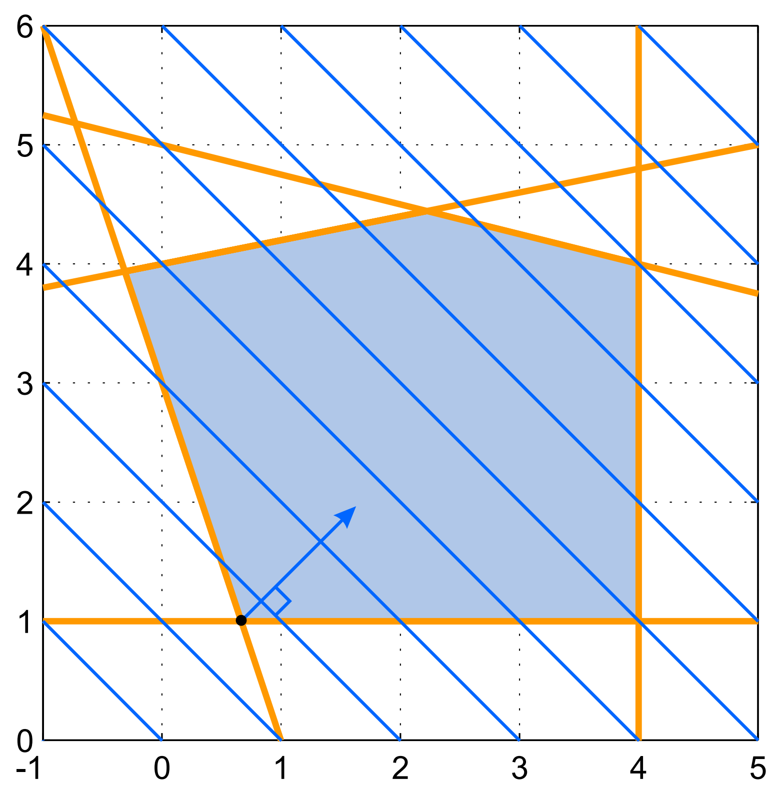
Linear Programming (LP)

In a *linear program*, the objective and constraint functions are affine.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & Cx \leq d \end{aligned}$$

Example

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \geq 3 \\ & x_2 \geq 1 \\ & x_1 \leq 4 \\ & -x_1 + 5x_2 \leq 20 \\ & x_1 + 4x_2 \leq 20 \end{aligned}$$



Linear Programming

Every linear program may be written in the *standard primal form*

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Here $x \in \mathbb{R}^n$, and $x \geq 0$ means $x_i \geq 0$ for all i

- The *nonnegative orthant* $\{ x \in \mathbb{R}^n \mid x \geq 0 \}$ is a *convex cone*.
- This convex cone defines the partial ordering $\geq$ on $\mathbb{R}^n$
- Geometrically, the feasible set is the intersection of an affine set with a convex cone.
- Any closed convex cone K with nonempty interior defines a partial ordering; the dual cone is

$$K^* = \{ y \mid x^T y \geq 0 \text{ for all } x \in K \}$$

Semidefinite Programming

$$\begin{aligned} \text{minimize} \quad & \mathbf{trace}\, CX \\ \text{subject to} \quad & \mathbf{trace}\, A_i X = b_i \qquad \text{for all } i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

- The variable X is in the set of $n \times n$ symmetric matrices

$$\mathbb{S}^n = \{ A \in \mathbb{R}^{n \times n} \mid A = A^T \}$$

- $X \succeq 0$ means X is positive semidefinite
- As for LP, the feasible set is the intersection of an affine set with a convex cone, in this case the *positive semidefinite cone*

$$\{ X \in \mathbb{S}^n \mid X \succeq 0 \}$$

Hence the feasible set is convex.

SDPs with Explicit Variables

We can also explicitly parametrize the affine set to give

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *linear matrix inequality*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

The Feasible Set is Semialgebraic

The *(basic closed) semialgebraic set* defined by polynomials $f_1, \ldots, f_m$ is

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \right\}$$

The feasible set of an SDP is a semialgebraic set.

Because a matrix $A \succ 0$ if and only if

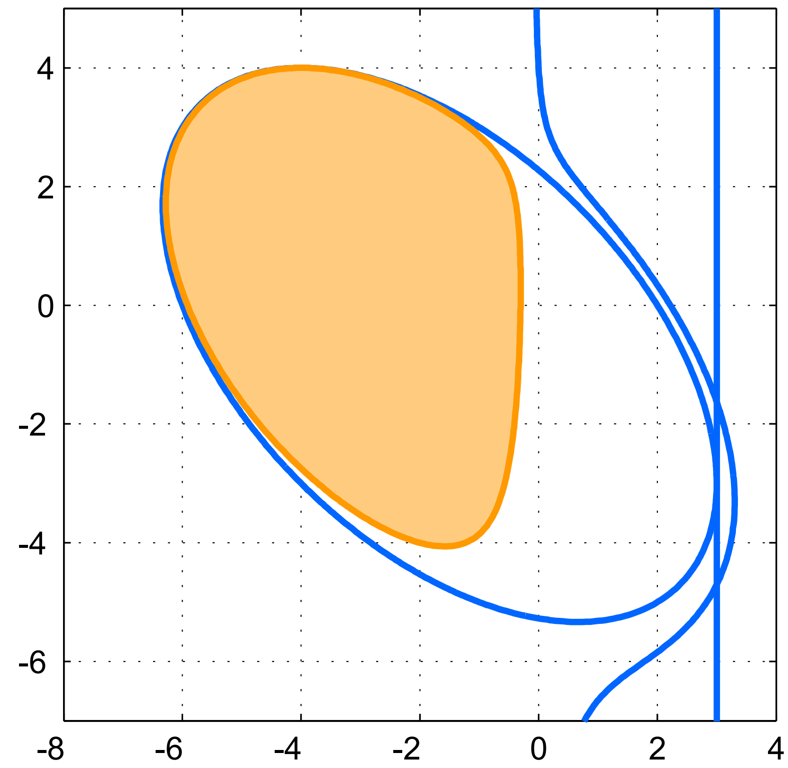
$$\det(A_k) > 0 \text{ for } k = 1, \ldots, n$$

where A_k is the submatrix of A consisting of the first k rows and columns.

The Feasible Set

For example

$$0 \prec \begin{bmatrix} 3-x_1 & -(x_1+x_2) & 1 \\ -(x_1+x_2) & 4-x_2 & 0 \\ 1 & 0 & -x_1 \end{bmatrix}$$



is equivalent to the polynomial inequalities

$$0 < 3 - x_1$$

$$0 < (3-x_1)(4-x_2) - (x_1+x_2)^2$$

$$0 < -x_1((3-x_1)(4-x_2) - (x_1+x_2)^2) - (4-x_2)$$

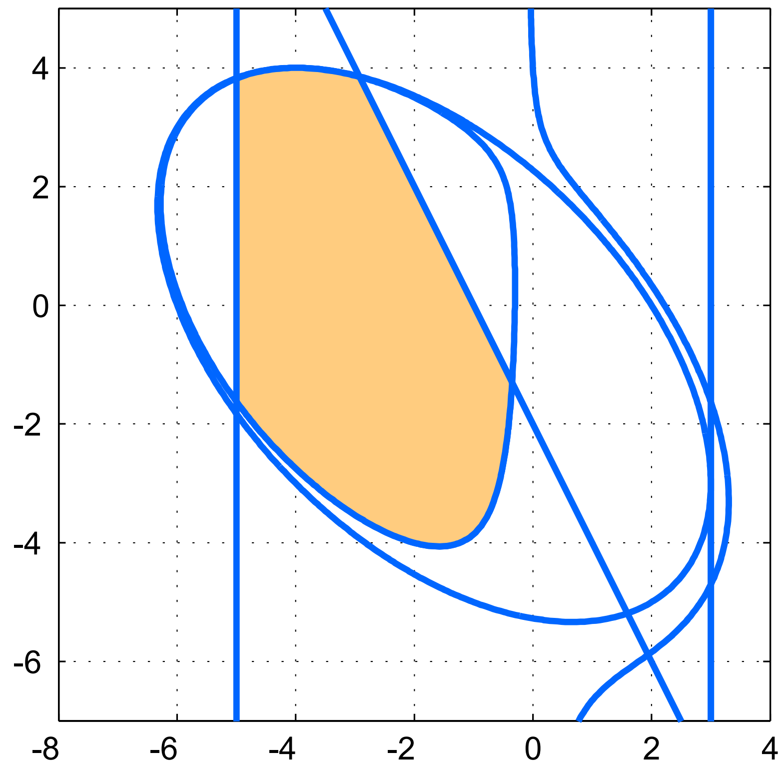
Intersection of Feasible Sets

The intersection of the feasible sets

$$\begin{bmatrix} 2x_1 + x_2 + 2 & 0 \\ 0 & -x_1 - 5 \end{bmatrix} \prec 0$$

and

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \prec 0$$

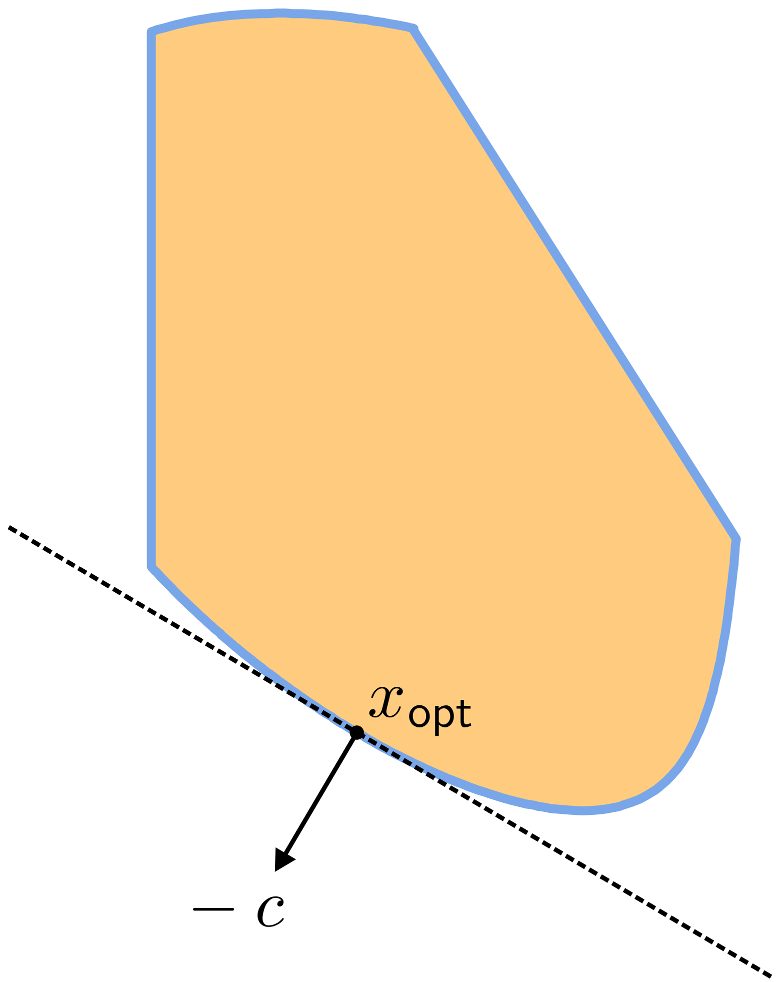


is given by

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 & 0 & 0 \\ x_1 + x_2 & x_2 - 4 & 0 & 0 & 0 \\ -1 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 2x_1 + x_2 + 2 & 0 \\ 0 & 0 & 0 & 0 & -x_1 - 5 \end{bmatrix} \prec 0$$

Optimal Points

Since SDPs are convex, if the feasible set is closed then the optimal is always achieved on the boundary.



LMIs with Matrix Variables

The inequality

$$\begin{bmatrix} A^TY + Y^TA & Y^TB \\ B^TY & -I \end{bmatrix} < 0$$

is a linear matrix inequality in the variable Y. Here

$$A \in \mathbb{R}^{n\times n} \qquad B \in \mathbb{R}^{n\times m} \qquad Y \in \mathbb{R}^{n\times n}$$

This form is accepted by some software, e.g. YALMIP

To convert this to standard form, write

$$Y(x) = \begin{bmatrix} x_1 & x_{n+1} & \dots & \\ x_2 & & & \\ \vdots & & & \\ x_n & x_{2n} & \dots & x_{n^2} \end{bmatrix} \qquad \text{equivalently } x = \mathbf{vec}(Y)$$

Then $\begin{bmatrix} A^TY(x) + Y^T(x)A & Y^T(x)B \\ B^TY(x) & -I \end{bmatrix}$ is affine in x.

Convex Optimization Problems

For a convex optimization problem, the *feasible set*

$$S = \{ \, x \in \mathbb{R}^n \mid f_i(x) \leq 0 \text{ and } h_j(x) = 0 \text{ for all } i, j \, \}$$

is convex. So we can write the problem as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in S \end{array}$$

This approach emphasizes the *geometry* of the problem.

For a convex optimization problem, any local minimum is also a global minimum.

Feasibility Problems

We are also interested in *feasibility problems* as follows. Does there exist $x \in \mathbb{R}^n$ which satisfies

$$\begin{aligned} f_i(x) &\leq 0 && \text{for all } i = 1, \ldots, m \\ h_i(x) &= 0 && \text{for all } i = 1, \ldots, p \end{aligned}$$

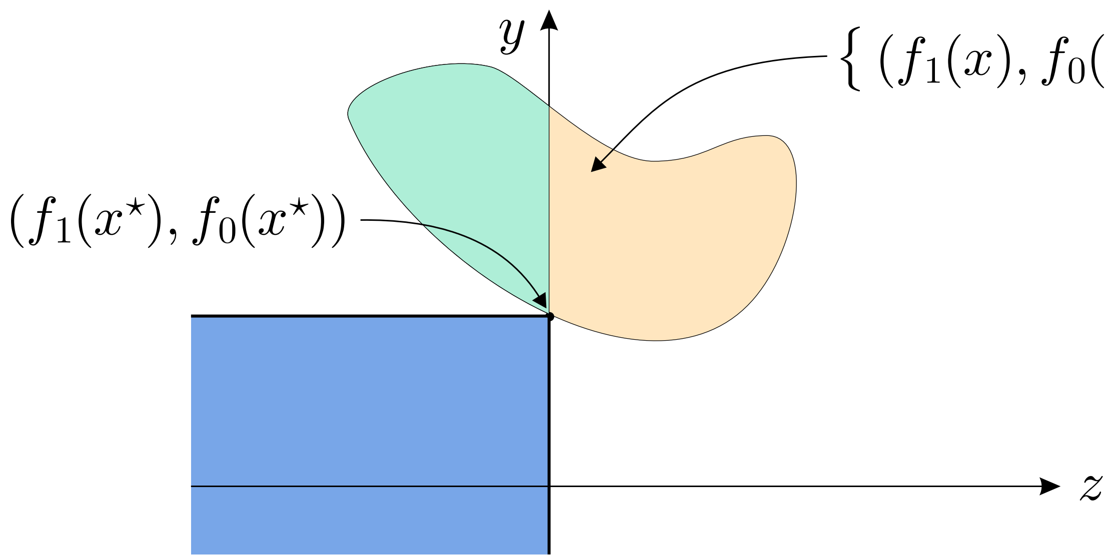
If there does not exist such an x, the problem is described as *infeasible*.

Feasibility Problems

We can always convert an optimization problem into a feasibility problem; does there exist $x \in \mathbb{R}^n$ such that

$$\begin{aligned} f_0(x) &\le t \\ f_i(x) &\le 0 \\ h_i(x) &= 0 \end{aligned}$$

Bisection search over the parameter t finds the optimal.



Feasibility Problems

Conversely, we can convert feasibility problems into optimization problems.

e.g. the feasibility problem of finding x such that

$$f_i(x) \leq 0 \qquad \text{for all } i = 1, \ldots, m$$

can be solved as

$$\begin{array}{rll} \text{minimize} & y & \\ \text{subject to} & f_i(x) \leq y & \text{for all } i = 1, \ldots, m \end{array}$$

where there are $n+1$ variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$

This technique may be used to find an initial feasible point for optimization algorithms

Algorithms

For convex optimization problems, there are several good algorithms

- interior-point algorithms work well in theory and practice
- for certain classes of problems, (e.g. LP and SDP) there is a worst-case time-complexity bound
- conversely, some convex optimization problems are known to be NP-hard
- problems are specified either in *standard form*, for LPs and SDPs, or via an *oracle*

Some Matlab Software

- *SeDuMi* `http://fewcal.kub.nl/sturm/software/sedumi.html`
- *YALMIP* `http://www.control.isy.liu.se/~johanl/yalmip.html`

Certificates

Consider the feasibility problem

Does there exist $x \in \mathbb{R}^n$ which satisfies

$$f_i(x) \leq 0 \ \text{ for all } i = 1, \ldots, m$$
$$h_i(x) = 0 \ \text{ for all } i = 1, \ldots, p$$

There is a fundamental asymmetry between establishing that

- There exists at least one feasible x
- The problem is infeasible

To show existence, one needs a *feasible point* $x \in \mathbb{R}^n$.

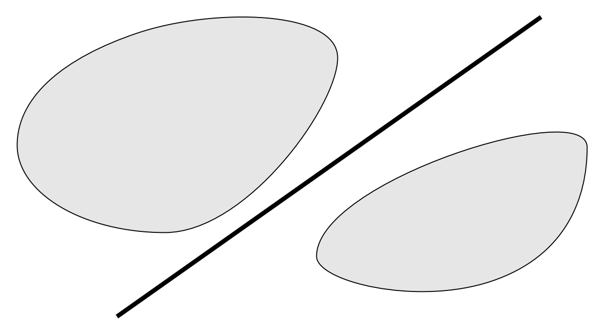
To show emptiness, one needs a a *certificate of infeasibility*; a mathematical proof that the problem is infeasible.

Certificates and Separating Hyperplanes

The simplest form of certificate is a *separating hyperplane*. The idea is that a hyperplane $L \subset \mathbb{R}^n$ breaks $\mathbb{R}^n$ into two half-spaces,

$$H_1 = \left\{ x \in \mathbb{R}^n \mid b^T x \leq a \right\} \qquad \text{and} \qquad H_2 = \left\{ x \in \mathbb{R}^n \mid b^T x > a \right\}$$

If two *closed, bounded and convex* sets are disjoint, there is a hyperplane that separates them.



So to prove infeasibility of

$$f_i(x) \leq 0 \qquad \text{for } i = 1, 2$$

we show that

$$\{ x \in \mathbb{R}^n \mid f_1(x) \leq 0 \} \subset H_1 \qquad \text{and} \qquad \{ x \in \mathbb{R}^n \mid f_2(x) \leq 0 \} \subset H_2$$

Duality

We'd like to solve

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \dots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \dots, p \end{array}$$

define the *Lagrangian* for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ by

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

and the *Lagrange dual function*

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu)$$

We allow $g(\lambda, \nu) = -\infty$ when there is no finite infimum

Duality

the *dual problem* is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

we call λ, ν *dual feasible* if $\lambda \geq 0$ and $g(\lambda, \nu)$ is finite.

- The dual function g is always concave, even if the primal problem is not convex

Weak Duality

For any primal feasible x and dual feasible λ, ν we have

$$\boxed{g(\lambda, \nu) \leq f_0(x)}$$

because

$$\begin{aligned} g(\lambda, \nu) &\leq f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ &\leq f_0(x) \end{aligned}$$

- A feasible λ, ν provides a *certificate* that the primal optimal is greater than $g(\lambda, \nu)$
- many interior-point methods simultaneously optimize the primal and the dual problem; when $f_0(x) - g(\lambda, \nu) \leq \varepsilon$ we know that x is $\varepsilon-$suboptimal

Strong Duality

- $p^\star$ is the optimal value of the primal problem,
- $d^\star$ is the optimal value of the dual problem

Weak duality means $p^\star \geq d^\star$

If $p^\star = d^\star$ we say *strong duality* holds. Equivalently, we say the *duality gap* $p^\star - d^\star$ is zero.

Constraint qualifications give sufficient conditions for strong duality.

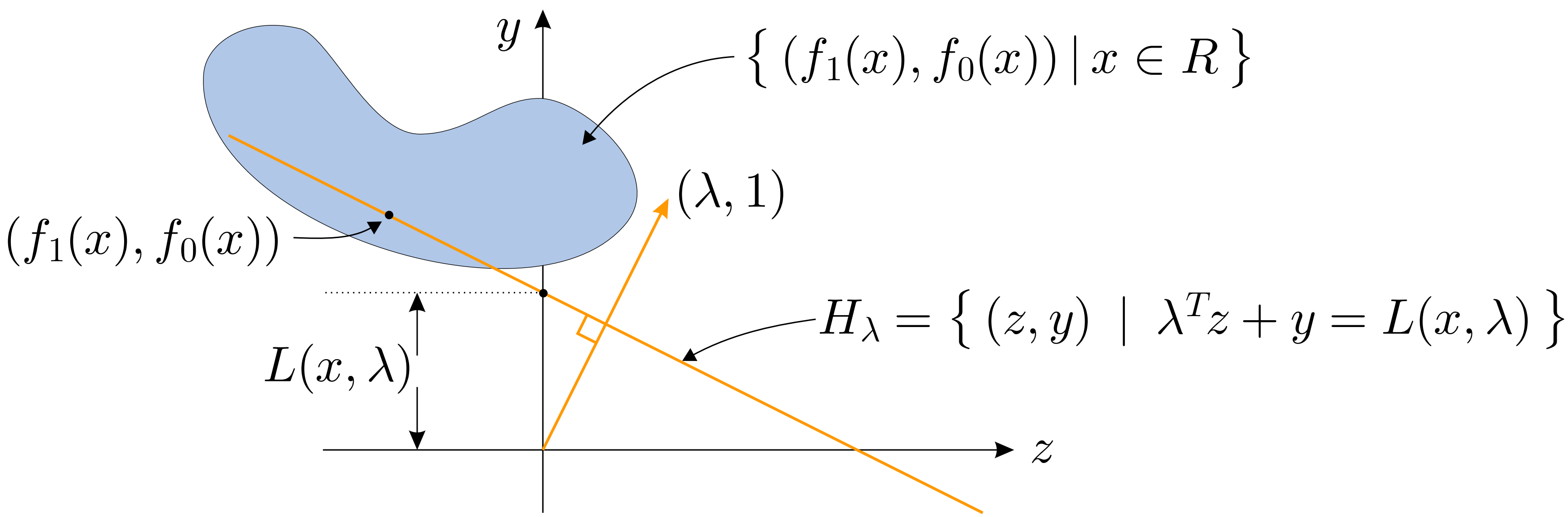
An example is *Slater's condition*; strong duality holds if the primal problem is convex and strictly feasible.

Geometric Interpretations: The Lagrangian

consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0 \end{array}$$

The value of the Lagrangian $L(x, \lambda)$ is the intersection of the hyperplane H_λ with the vertical axis

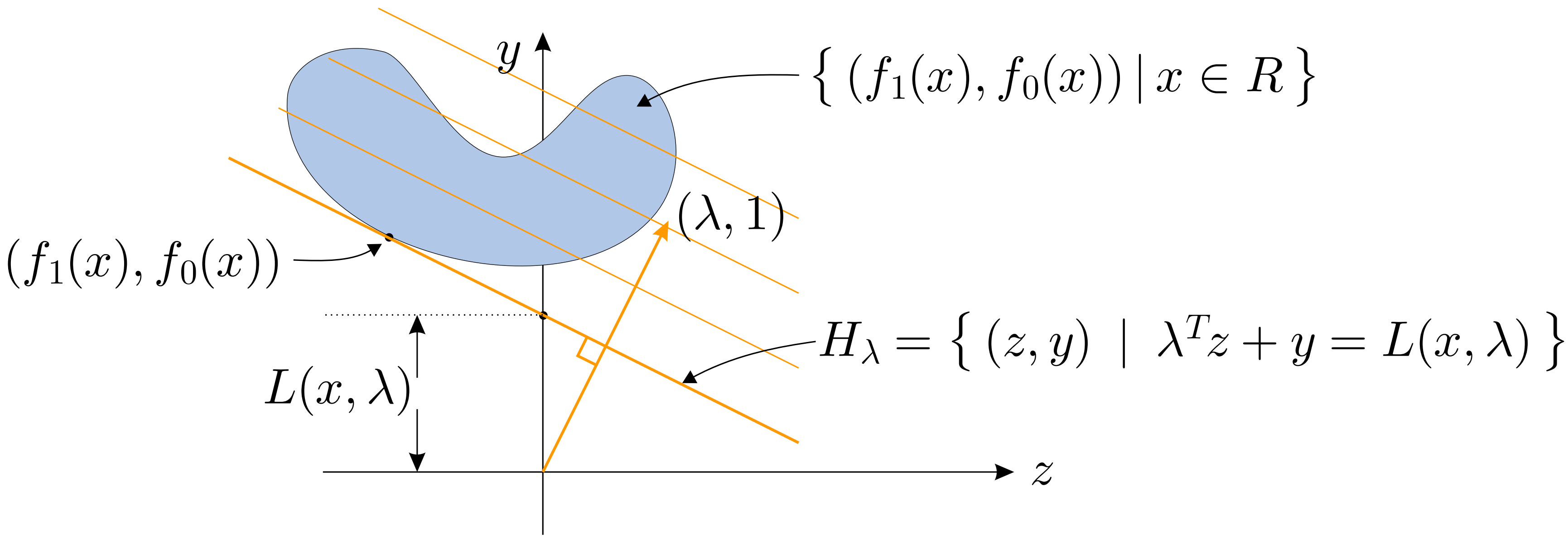


The Lagrange Dual Function

The Lagrange dual function is

$$g(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$$

i.e., the minimum intersection for a given slope $-\lambda$



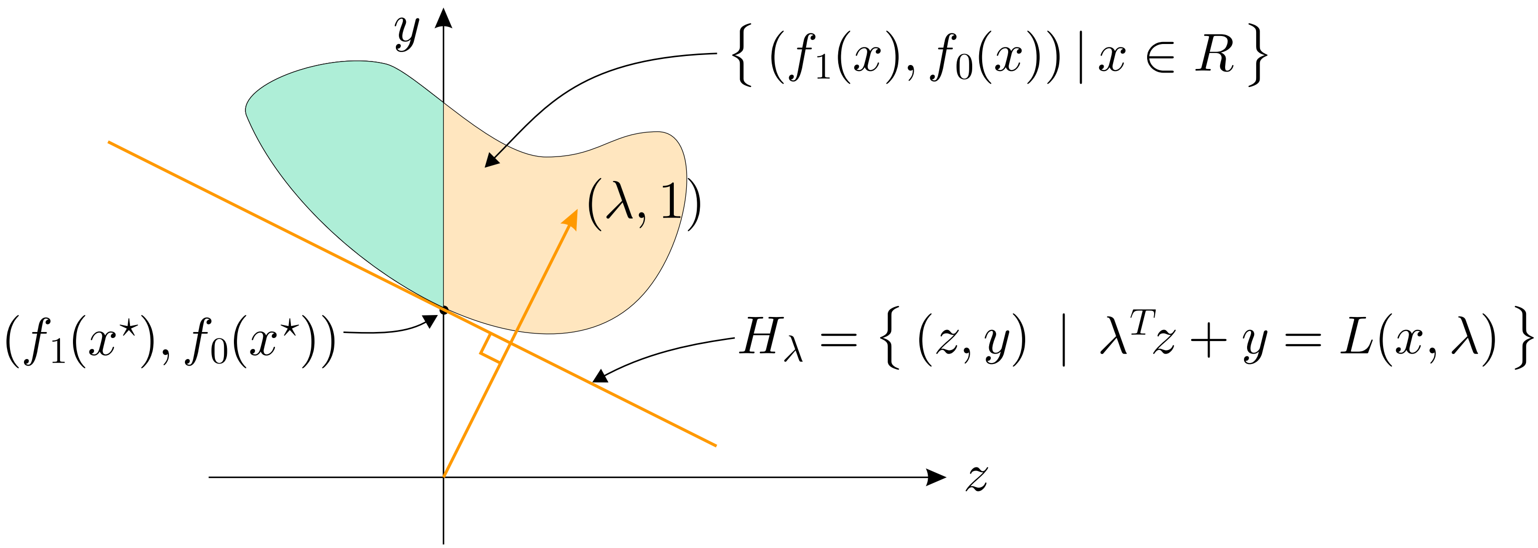
Sensitivity

consider the perturbed problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq y_i \quad \text{for all } i = 1, \dots, m \end{array}$$

and let $p^\star(y)$ be the optimal value parametrized by y. Then for any optimal $\lambda^\star$ we have

$$\lambda^\star = -\nabla p^\star(0)$$

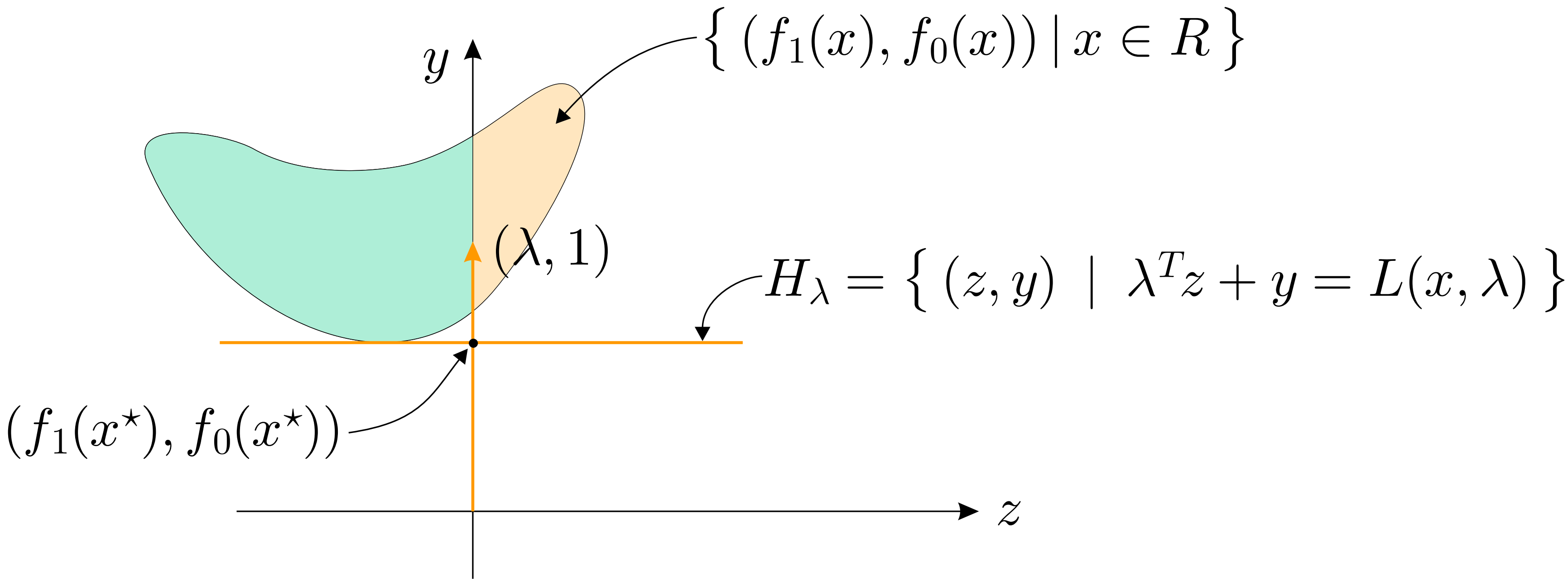


Complementary Slackness

For $\lambda^\star$ dual optimal, and $x^\star$ primal optimal, we have

$$\lambda_i^\star f_i(x^\star) = 0 \qquad \text{for all } i = 1, \ldots, m$$

whenever strong duality holds; i.e., if the i'th constraint is active, then $\lambda_i^\star = 0$



Example: Linear Programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

The Lagrange dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbb{R}^n} \left(c^T x + \nu^T (b - Ax) - \lambda^T x \right) \\ &= \begin{cases} b^T \nu & \text{if } c - A^T \nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

So the dual problem is

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & A^T \nu \leq c \end{array}$$

Example: Semidefinite Programming

$$
\begin{array}{lll}
\text{minimize} & \mathbf{trace}\, CX & \\
\text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\
 & X \succeq 0 &
\end{array}
$$

The Lagrange dual is

$$
\begin{aligned}
g(Z, \nu) &= \inf_X \left(\mathbf{trace}\, CX - \mathbf{trace}\, ZX + \sum_{i=1}^m \nu_i (b_i - \mathbf{trace}\, A_i X) \right) \\
&= \begin{cases} b^T \nu & \text{if } C - Z - \sum_{i=1}^m \nu_i A_i = 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

So the dual problem is to maximize $b^T \nu$ subject to

$$
C - Z - \sum_{i=1}^m \nu_i A_i = 0 \qquad \text{and} \qquad Z \succeq 0
$$

Semidefinite Programming Duality

The primal problem is

$$\begin{array}{lll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} \nu_i A_i \preceq C \end{array}$$

The Fourfold Way

There are several ways of formulating an SDP for its numerical solution.

Because *subspaces* can be described

- Using *generators* or a *basis*; Equivalently, the subspace is the range of a linear map $\{\, x \mid x = B\lambda \text{ for some } \lambda \,\}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 - \lambda_2 \\ 2\lambda_1 + 2\lambda_2 \end{bmatrix}$$

- Through the defining equations; i.e, as the *kernel* $\{\, x \mid Ax = 0 \,\}$

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \quad | \quad 4x_1 - 2x_2 - x_3 = 0\}$$

Depending on which description we use, and whether we write a primal or dual formulation, we have *four* possibilities (two primal-dual pairs).

Example: Two Primal-Dual Pairs

$$\begin{array}{ll} \text{maximize} & 2x + 2y \\ \text{subject to} & \begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix} \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & \mathbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} W \\ \text{subject to} & \mathbf{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} W = 2 \\ & \mathbf{trace} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} W = 2 \\ & W \succeq 0 \end{array}$$

Another, *more efficient* fomulation which solves the same problem:

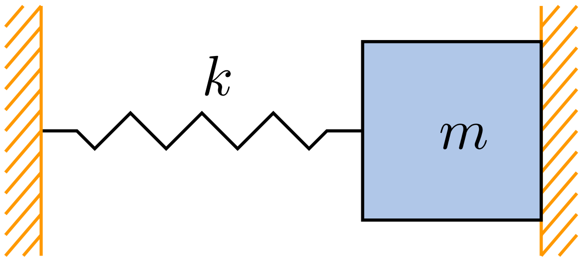
$$\begin{array}{ll} \text{maximize} & \mathbf{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} Z \\ \text{subject to} & \mathbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z = 2 \\ & Z \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2t \\ \text{subject to} & \begin{bmatrix} t-1 & -1 \\ -1 & t+1 \end{bmatrix} \succeq 0 \end{array}$$

Duality

- Duality has many interpretations; via economics, game-theory, geometry.
- e.g., one may interpret Lagrange multipliers as a price for violating constraints, which may correspond to resource limits or capacity constraints.
- Often physical problems associate specific meaning to certain Lagrange multipliers, e.g. pressure, momentum, force can all be viewed as Lagrange multipliers

Example: Mechanics



- Spring under compression
- Mass at horizontal position x, equilibrium at $x = 2$

$$\begin{array}{ll} \text{minimize} & \frac{k}{2}(x-2)^2 \\ \text{subject to} & x \le 1 \end{array}$$

The Lagrangian is $L(x, \lambda) = \frac{k}{2}(x-2)^2 + \lambda(x-1)$

If λ is dual optimal and x is primal optimal, then $\frac{\partial}{\partial x} L(x, \lambda) = 0$, i.e.,

$$k(x-2) + \lambda = 0$$

so we can interpret λ as a *force*

Feasibility of Inequalities

The *primal feasibility problem* is

> does there exist $x \in \mathbb{R}^n$ such that
> $f_i(x) \geq 0 \qquad$ for all $i = 1, \dots, m$

The *dual function* $g : \mathbb{R}^m \to \mathbb{R}$ is

$$g(\lambda) = \sup_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \lambda_i f_i(x)$$

The *dual feasibility problem* is

> does there exist $\lambda \in \mathbb{R}^m$ such that
> $$g(\lambda) < 0$$
> $$\lambda \geq 0$$

Theorem of Alternatives

If the dual problem is feasible, then the primal problem is infeasible.

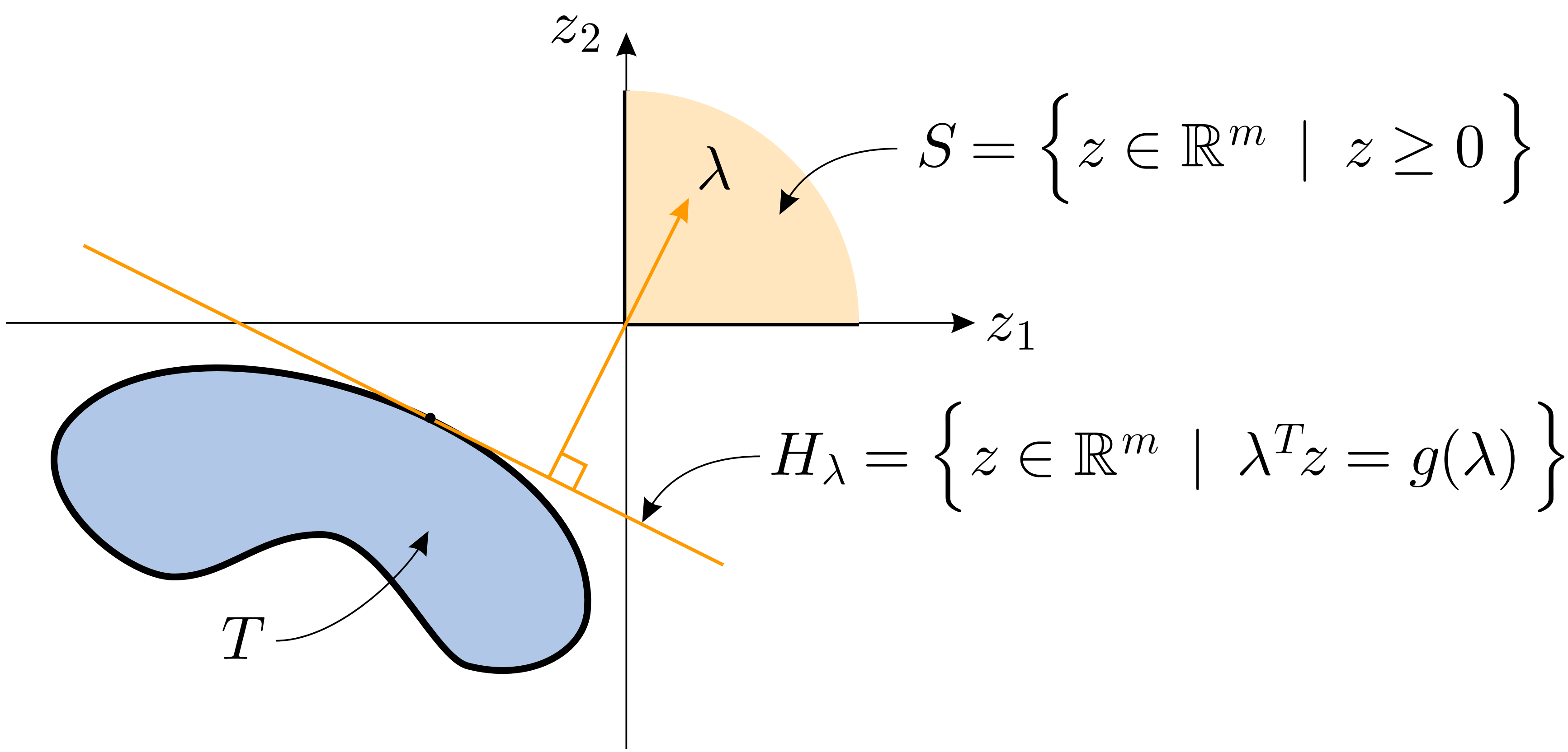
Proof

Suppose the primal problem is feasible, and let $\tilde{x}$ be a feasible point. Then

$$
\begin{aligned}
g(\lambda) &= \sup_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \lambda_i f_i(x) \\
&\geq \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) \qquad \text{for all } \lambda \in \mathbb{R}^m
\end{aligned}
$$

and so $g(\lambda) \geq 0$ for all $\lambda \geq 0$.

Geometric Interpretation



if $g(\lambda) < 0$ and $\lambda \geq 0$ then the hyperplane H_λ separates S from T, where

$$T = \left\{ \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$$

Certificates

- A dual feasible point gives a *certificate* of infeasibility of the primal problem.
- If the Lagrange dual function g is easy to compute, and we can show $g(\lambda) < 0$, then this is a *proof* that the primal is infeasible.
- One way to do this is to have an explicit expression for

$$g(\lambda) = \sup_x L(x, \lambda)$$

  where for feasibility problems, the Lagrangian is $L(x, \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x)$

- Alternatively, given λ, we may be able to show directly that

$$L(x, \lambda) < -\varepsilon \qquad \text{for all } x \in \mathbb{R}^n$$

  for some $\varepsilon > 0$.