9. Elimination

- Ideal membership
- Solving polynomial equations
- Elimination and implicitization
- The elimination ideal
- The elimination theorem
- Geometric interpretation
ideal membership

given \( h, f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_m] \), we’d like to test if

\[
h \in \text{ideal}\{f_1, \ldots, f_m\}
\]

procedure

- compute the Groebner basis \( g_1, \ldots, g_s \) for \( f_1, \ldots, f_m \)
- divide \( h \) by \( g_1, \ldots, g_s \); the remainder \( r = 0 \) if and only if

\[
h \in \text{ideal}\{f_1, \ldots, f_m\}
\]

this works independent of the monomial order or the order in which division is performed.
example

for $f_1 = xz - y^2$, $f_2 = x^3 - z^2$ in grlex order, the Groebner basis is

$$xz - y^2, \quad x^3 - z^2, \quad x^2y^2 - z^3, \quad xy^4 - z^4, \quad y^6 - z^5$$

check membership of $h = -4x^2y^2z^2 + y^6 + 3z^5$, we find

$$h = (-4xy^2z - 4y^4)(xz - y^2) + (-3)(y^6 - z^5)$$

so $h \in \text{ideal}\{f_1, f_2\}$

also if $t = xy - 5z^2 + x$, then $t$ is not in the ideal, since its leading term is not divisible by any of the leading terms of the Groebner basis
example: solving polynomial equations

consider the equations

\[ x^2 + y^2 + z^2 - 1 = 0 \]
\[ x^2 - y + z^2 = 0 \]
\[ x - z = 0 \]

a Groebner basis in \textit{lex order} gives equivalent equations

\[ x - z = 0 \]
\[ y - 2z^2 = 0 \]
\[ 4z^4 + 2z^2 - 1 = 0 \]

the third equation depends only on \( z \); so we can solve it, then substitute to find \( x \) and \( y \)
example 2

we’d like to solve the following equations

\[-2w x + 3x^2 + 2yz = 0\]
\[-2wy + 2xz = 0\]
\[-2wz + 2xy - 2z = 0\]
\[x^2 + y^2 + z^2 - 1 = 0\]
example 2 continued

a Groebner basis in lex order \( w > x > y > z \) gives equivalent equations

\[
\begin{align*}
7670 w - 11505 x - 11505 y z & - 335232 z^6 + 477321 z^4 - 134419 z^2 = 0 \\
x^2 + y^2 + z^2 - 1 & = 0 \\
3835 x y & - 19584 z^5 + 25987 z^3 - 6403 z = 0 \\
-3835 x z & - 3835 y z^2 + 1152 z^5 + 1404 z^3 - 2556 z = 0 \\
-3835 y^3 & - 3835 y z^2 + 3835 y + 9216 z^5 - 11778 z^3 + 2562 z = 0 \\
3835 y^2 z & - 6912 z^5 + 10751 z^3 - 3839 z = 0 \\
118 y z^3 & - 118 y z - 1152 z^6 + 1605 z^4 - 453 z^2 = 0 \\
-1152 z^7 & + 1763 z^5 - 655 z^3 + 44 z = 0
\end{align*}
\]

again, the Groebner basis eliminates variables successively similar to back-substitution in Gaussian elimination
Elimination

• the above examples illustrate *elimination*

• the Groebner basis algorithm successively removes terms
  this is similar to Gaussian elimination; a *triangular* structure results, i.e.,

  some polynomials depend only on $x_n$
  some polynomials depend only on $x_{n-1}, x_n$
  some polynomials depend only on $x_{n-2}, x_{n-1}, x_n$
  etc.
Implicitization

A parametrization of the circle is

\[ x = \frac{1 - t^2}{1 + t^2}, \]
\[ y = \frac{2t}{1 + t^2}. \]

clear denominators

\[ t^2 y - 2t + y = 0 \]
\[ t^2 x + t^2 + x - 1 = 0 \]

Groebner basis in lex order \( t > x > y \) is

\[ tx + t - y \]
\[ ty + x - 1 \]
\[ x^2 + y^2 - 1 \]

so for any \( t \neq 0 \) every \((x, y)\) lies on the circle
elimination

- if \( \{f_1, \ldots, f_m\} \) and \( \{g_1, \ldots, g_m\} \) are two bases for the same ideal, then they have the same feasible sets

- in particular, in the above example, this implies that every solution to the implicit equations satisfies

\[
x^2 + y^2 = 1
\]

that is \( \mathcal{V}\{f_1, \ldots, f_m\} \subset \mathcal{V}\{g_3\} \)

- but the set on the RHS is strictly bigger; it contains \((-1, 0)\)

- because we have ignored \(g_1\) and \(g_2\)
the elimination ideal

the Groebner basis $G = \{g_1, \ldots, g_m\}$ w.r.t. lex order consists of

- polynomials in $I = \text{ideal}\{g_1, \ldots, g_m\}$
- which do not contain variables $x_1, \ldots, x_k$ for some $k$

that is, it finds polynomials in

$$I_k = \text{ideal}\{g_1, \ldots, g_m\} \in \mathbb{K}[x_{k+1}, \ldots, x_n]$$

- $I_k$ is called the $k$'th elimination ideal of $I$
- it is an ideal in $\mathbb{K}[x_{k+1}, \ldots, x_n]$
- every $f \in I_k$ is a polynomial consequence of $g_1, \ldots, g_m$
  which depends only on $x_{k+1}, \ldots, x_n$
the elimination theorem

suppose $G = \{g_1, \ldots, g_m\}$ is a Groebner basis for $I$ w.r.t. lex order with $x_1 > x_2 > \cdots > x_n$; then

$$G_k = G \cap \mathbb{K}[x_{k+1}, \ldots, x_n]$$

is a Groebner basis for $I_k = I \cap \mathbb{K}[x_{k+1}, \ldots, x_n]$.

we need to show

$$\text{ideal}\{\text{lt}(I_k)\} = \text{ideal}\{\text{lt}(G_k)\}$$

since $I_k \supset G_k$, all we need to show is LHS $\subset$ RHS.

any $f \in I_k$ is divisible by $\text{lt}(g_i)$ for some $g_i$, and $f$ does not contain variables $x_{k+1}, \ldots, x_n$, so neither does $\text{lt}(g_i)$;

since we are using lex order, neither does $g_i$, so $g_i \in G_k$.
example

consider polynomials \( x^2 + y + z - 1 \), \( x + y^2 + z - 1 \), \( x + y + z^2 - 1 \)

Groebner basis is

\[
\begin{align*}
g_1 &= x + y + z^2 - 1 \\
g_2 &= y^2 - y - z^2 + z \\
g_3 &= 2y^2 + z^4 - z^2 \\
g_4 &= z^6 - 4z^4 + 4z^3 - z^2
\end{align*}
\]

so we have

\[
I_1 = I \cap \mathbb{K}[y, z] = \text{ideal}\{g_2, g_3, g_4\}
\]

\[
I_2 = I \cap \mathbb{K}[z] = \text{ideal}\{g_4\}
\]

- \( I_{n-1} \) is always principal
- any polynomial in \( I \) which does not contain \( x, y \) is a multiple of \( g_4 \)
geometric interpretation

in parametrization or elimination, we are interested in

\[
\left\{ (x_{k+1}, \ldots, x_n) \mid \text{there exists } x_1, \ldots, x_k \text{ such that } x \in \mathcal{V}\{f_1, \ldots, f_m\} \right\}
\]

this is the projection of \( \mathcal{V}(f_1, \ldots, f_m) \) onto \( x_1 = 0, \ldots, x_k = 0 \)

denote the projection map by

\[
P_k : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}
\]

\[
x \mapsto (0, \ldots, 0, x_{k+1}, \ldots, x_n)
\]

we have

\[
P_k \mathcal{V}(I) \subseteq \mathcal{V}(I_k)
\]
projection

Suppose $I$ is an ideal, and $I_k$ is the $k$'th elimination ideal; then

$$P_k \mathcal{V}(I) \subset \mathcal{V}(I_k)$$

Because if $f \in I_k$ then $f(x) = 0$ for all $x \in \mathcal{V}(I)$

But since $f$ doesn’t depend on $x_1, \ldots, x_k$,

$$f(P_k x) = 0 \quad \text{for all} \quad x \in \mathcal{V}(I)$$

Which means

$$f(y) = 0 \quad \text{for all} \quad y \in P_k \mathcal{V}(I)$$