9. Elimination

- Ideal membership
- Solving polynomial equations
- Elimination and implicitization
- The elimination ideal
- The elimination theorem
- Geometric interpretation

ideal membership

given $h, f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_m]$, we'd like to test if

$$h \in \textbf{ideal}\{f_1, \ldots, f_m\}$$

procedure

- compute the Groebner basis $g_1, \ldots, g_s$ for $f_1, \ldots, f_m$
- divide h by $g_1, \ldots, g_s$; the remainder $r = 0$ if and only if

$$h \in \textbf{ideal}\{f_1, \ldots, f_m\}$$

this works independent of the monomial order or the order in which division is performed.

example

for $f_1 = x\,z - y^2$, $f_2 = x^3 - z^2$ in grlex order, the Groebner basis is

$$x\,z - y^2 \qquad x^3 - z^2 \qquad x^2\,y^2 - z^3 \qquad x\,y^4 - z^4 \qquad y^6 - z^5$$

check membership of $h = -4\,x^2\,y^2\,z^2 + y^6 + 3\,z^5$, we find

$$h = (-4\,x\,y^2\,z - 4\,y^4)(x\,z - y^2) + (-3)(y^6 - z^5)$$

so $h \in \mathbf{ideal}\{f_1, f_2\}$

also if $t = xy - 5z^2 + x$, then t is not in the ideal, since its leading term is not divisible by any of the leading terms of the Groebner basis

example: solving polynomial equations

consider the equations

$$x^2 + y^2 + z^2 - 1 = 0$$
$$x^2 - y + z^2 = 0$$
$$x - z = 0$$

a Groebner basis in *lex order* gives equivalent equations

$$x - z = 0$$
$$y - 2\,z^2 = 0$$
$$4\,z^4 + 2\,z^2 - 1 = 0$$

the third equation depends only on z; so we can solve it, then substitute to find x and y

example 2

we'd like to solve the following equations

$$
\begin{aligned}
-2\,w\,x + 3\,x^2 + 2\,y\,z &= 0 \\
-2\,w\,y + 2\,x\,z &= 0 \\
-2\,w\,z + 2\,x\,y - 2\,z &= 0 \\
x^2 + y^2 + z^2 - 1 &= 0
\end{aligned}
$$

example 2 continued

a Groebner basis in lex order $w > x > y > z$ gives equivalent equations

$$7670\,w - 11505\,x - 11505\,y\,z - 335232\,z^6 + 477321\,z^4 - 134419\,z^2 = 0$$
$$x^2 + y^2 + z^2 - 1 = 0$$
$$3835\,x\,y - 19584\,z^5 + 25987\,z^3 - 6403\,z = 0$$
$$-3835\,x\,z - 3835\,y\,z^2 + 1152\,z^5 + 1404\,z^3 - 2556\,z = 0$$
$$-3835\,y^3 - 3835\,y\,z^2 + 3835\,y + 9216\,z^5 - 11778\,z^3 + 2562\,z = 0$$
$$3835\,y^2\,z - 6912\,z^5 + 10751\,z^3 - 3839\,z = 0$$
$$118\,y\,z^3 - 118\,y\,z - 1152\,z^6 + 1605\,z^4 - 453\,z^2 = 0$$
$$-1152\,z^7 + 1763\,z^5 - 655\,z^3 + 44\,z = 0$$

again, the Groebner basis eliminates variables successively

similar to back-substitution in Gaussian elimination

Elimination

- the above examples illustrate *elimination*

- the Groebner basis algorithm successively removes terms

  this is similar to Gaussian elimination; a *triangular* structure results, i.e,

  some polynomials depend only on x_n

  some polynomials depend only on x_{n-1}, x_n

  some polynomials depend only on x_{n-2}, x_{n-1}, x_n

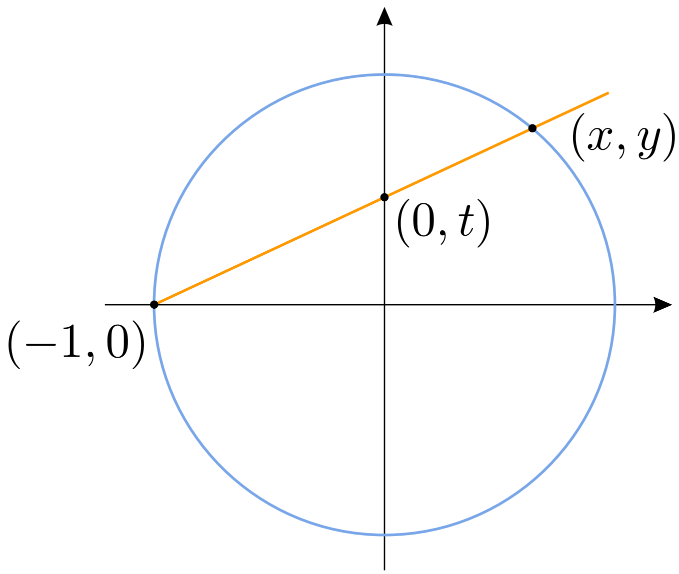
  etc.

Implicitization

a parametrization of the circle is

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$



clear denominators

$$t^2 y - 2t + y = 0 \qquad t^2 x + t^2 + x - 1 = 0$$

Groebner basis in lex order $t > x > y$ is

$$t x + t - y \qquad t y + x - 1 \qquad x^2 + y^2 - 1$$

so for any $t \neq 0$ every (x, y) lies on the circle

elimination

- if $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_m\}$ are two bases for the same ideal, then they have the same feasible sets

- in particular, in the above example, this implies that *every solution* to the implicit equations satisfies

$$x^2 + y^2 = 1$$

  that is $\mathcal{V}\{f_1, \ldots, f_m\} \subset \mathcal{V}\{g_3\}$

- but the set on the RHS is *strictly bigger*; it contains $(-1, 0)$

- because we have ignored g_1 and g_2

the elimination ideal

the Groebner basis $G = \{g_1, \dots, g_m\}$ w.r.t. lex order consists of

- polynomials in $I = \mathbf{ideal}\{g_1, \dots, g_m\}$
- which do not contain variables $x_1, \dots, x_k$ for some k

that is, it finds polynomials in

$$\boxed{I_k = \mathbf{ideal}\{g_1, \dots, g_m\} \in \mathbb{K}[x_{k+1}, \dots, x_n]}$$

- I_k is called the *k'th elimination ideal* of I
- it is an ideal in $\mathbb{K}[x_{k+1}, \dots, x_n]$
- every $f \in I_k$ is a *polynomial consequence* of $g_1, \dots, g_m$ which depends only on $x_{k+1}, \dots, x_n$

the elimination theorem

suppose $G = \{g_1, \ldots, g_m\}$ is a Groebner basis for I w.r.t. lex order with $x_1 > x_2 > \cdots > x_n$; then

$$\boxed{G_k = G \cap \mathbb{K}[x_{k+1}, \ldots, x_n]}$$

is a Groebner basis for $I_k = I \cap \mathbb{K}[x_{k+1}, \ldots, x_n]$

we need to show

$$\mathbf{ideal}\{\mathbf{lt}(I_k)\} = \mathbf{ideal}\{\mathbf{lt}(G_k)\}$$

since $I_k \supset G_k$, all we need to show is LHS $\subset$ RHS

any $f \in I_k$ is divisible by $\mathbf{lt}(g_i)$ for some g_i, and f does not contain variables $x_{k+1}, \ldots, x_n$, so neither does $\mathbf{lt}(g_i)$;
since we are using lex order, neither does g_i, so $g_i \in G_k$

example

consider polynomials $x^2 + y + z - 1,\ x + y^2 + z - 1,\ x + y + z^2 - 1$

Groebner basis is

$$g_1 = x + y + z^2 - 1 \qquad g_2 = y^2 - y - z^2 + z$$
$$g_3 = 2\,y\,z^2 + z^4 - z^2 \qquad g_4 = z^6 - 4\,z^4 + 4\,z^3 - z^2$$

so we have

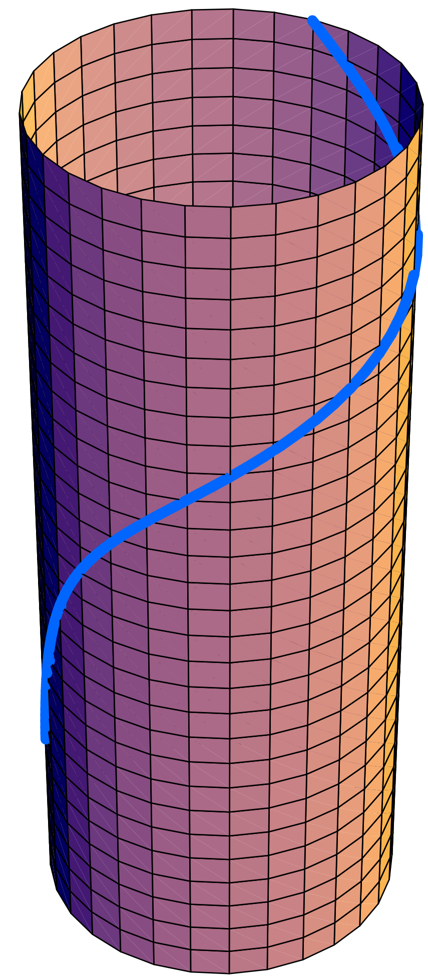
$$I_1 = I \cap \mathbb{K}[y, z] = \textbf{ideal}\{g_2, g_3, g_4\}$$
$$I_2 = I \cap \mathbb{K}[z] = \textbf{ideal}\{g_4\}$$

- I_{n-1} is always principal
- any polynomial in I which does not contain x, y is a multiple of g_4

geometric interpretation

in parametrization or elimination, we are interested in

$$\left\{ (x_{k+1}, \dots, x_n) \;\middle|\; \text{there exists } x_1, \dots, x_k \text{ such that } x \in \mathcal{V}\{f_1, \dots, f_m\} \right\}$$



this is the *projection* of $\mathcal{V}(f_1, \dots, f_m)$ onto $x_1 = 0, \dots, x_k = 0$

denote the projection map by

$$\begin{aligned} P_k : \mathbb{R}^n &\to \mathbb{R}^{n-k} \\ x &\mapsto (0, \dots, 0, x_{k+1}, \dots, x_n) \end{aligned}$$

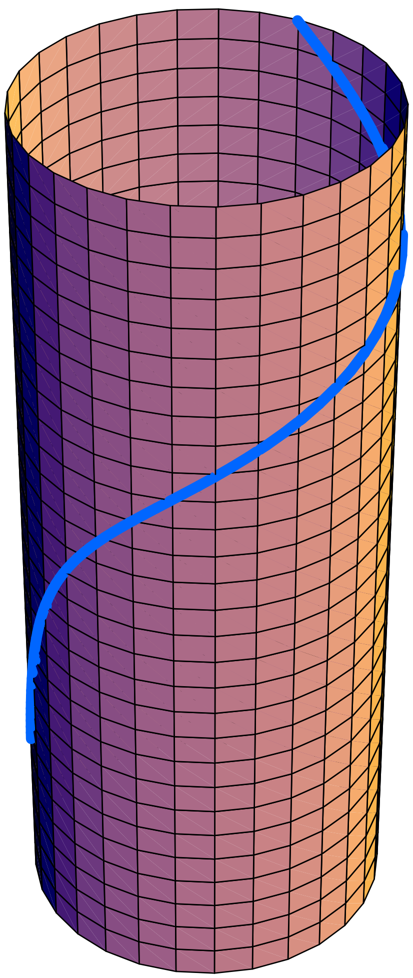
we have

$$\boxed{P_k \mathcal{V}(I) \subset \mathcal{V}(I_k)}$$

projection

suppose I is an ideal, and I_k is the k'th elimination ideal; then

$$P_k\mathcal{V}(I) \subset \mathcal{V}(I_k)$$



because if $f \in I_k$ then $f(x) = 0$ for all $x \in \mathcal{V}(I)$

but since f doesn't depend on $x_1, \ldots, x_k$,

$$f(P_k x) = 0 \qquad \text{for all} \qquad x \in \mathcal{V}(I)$$

which means

$$f(y) = 0 \qquad \text{for all} \qquad y \in P_k\mathcal{V}(I)$$