16. More Fourier-Motzkin Elimination

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efficiency of Fourier-Motzkin elimination

if A has m rows, then after elimination of x_1 we can have no more than

$$\left\lfloor \frac{m^2}{4} \right\rfloor$$

facets

- if $m/2$ inequalities have a positive coefficient of x_1, and $m/2$ have a negative coefficient, then FM constructs exactly $m^2/4$ new inequalities
- repeating this, eliminating d dimensions gives

  $$\left\lfloor \frac{m}{2} \right\rfloor^{2^n}$$

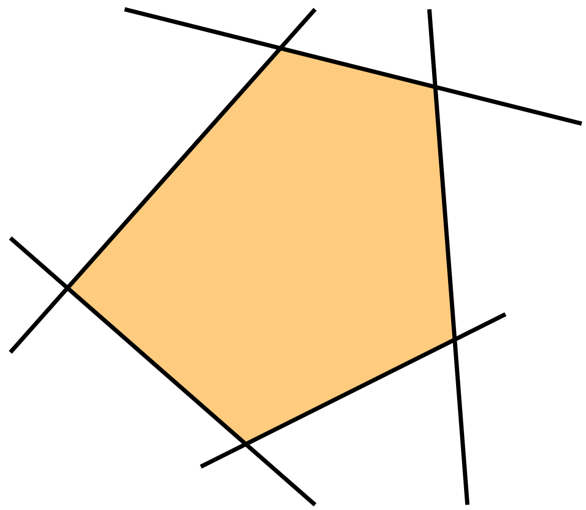
  inequalities
- key question: how may are are redundant? i.e., does projection produce exponentially more facets?

representation of polytopes

we can represent a polytope in the following ways

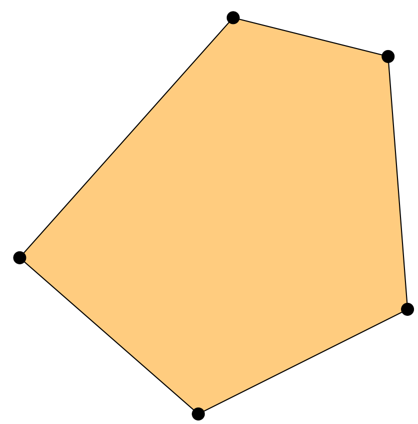
- *an intersection of halfspaces*, called an H-polytope

$$S = \Big\{ x \in \mathbb{R}^n \mid Ax \leq b \Big\}$$



- *the convex hull of its vertices*, called a V-polytope

$$S = \mathbf{co}\Big\{ a_1, \ldots, a_m \Big\}$$



size of representations

in some cases, one representation is smaller than the other

- the n-cube

$$C_n = \Big\{ x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \text{ for all } i \Big\}$$

  has $2n$ facets, and 2^n vertices

- the n-dimensional *crosspolytope*

$$\begin{aligned} C_n^* &= \Big\{ x \in \mathbb{R}^n \;\Big|\; \sum_i |x_i| \leq 1 \Big\} \\ &= \mathbf{co}\,\{ e_1, -e_1, \ldots, e_n, -e_n \} \end{aligned}$$

  has $2n$ vertices and 2^n facets

optimization problems

the *optimization problem*: given polytope S and $c \in \mathbb{R}^n$, find x that solves

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in S \end{array}$$

or state that $S = \emptyset$

roughly speaking, an equivalent problem (via bisection search) is *halfspace containment*

given $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, is it true that

$$S \subset \left\{ x \in \mathbb{R}^n \mid c^T x \leq \gamma \right\}$$

if not, find $x \in S$ such that $c^T x > \gamma$

membership problems

the *membership problem*: given polytope S and $y \in \mathbb{R}^n$, decide if $y \in S$, and if not find $\lambda \in \mathbb{R}^n$ such that

$$\lambda^T y > \max \{ \lambda^T x \mid x \in S \}$$

the membership problem is also called the *separation problem*

problem solving using different representations

- V-polytope: optimization is easy; evaluate $c^T x$ at all vertices

  for membership, we need to solve an LP; duality will give certificate of infeasibility

- H-polytope: membership is easy; evaluate $Ay - b$

  the certificate of infeasibility is just the violated inequality

  the optimization is an LP

converting between representations

suppose we are given a V-polytope

$$\begin{aligned} S &= \mathbf{co}\,\{\, a_1, \ldots, a_m \,\} \\ &= \left\{\, A^T\lambda \mid \lambda \geq 0,\ \lambda^T 1 = 1 \,\right\} \\ &= \left\{\, x \mid \text{there exists } \lambda \text{ such that } \lambda \geq 0,\ \lambda^T 1 = 1,\ x = A^T\lambda \,\right\} \end{aligned}$$

hence S is a *projection* onto $\lambda = 0$ of

$$\left\{\, (\lambda, x) \;\middle|\; \lambda \geq 0,\ \lambda^T 1 = 1,\ x = A^T\lambda \,\right\}$$

so we can use Fourier-Motzkin!

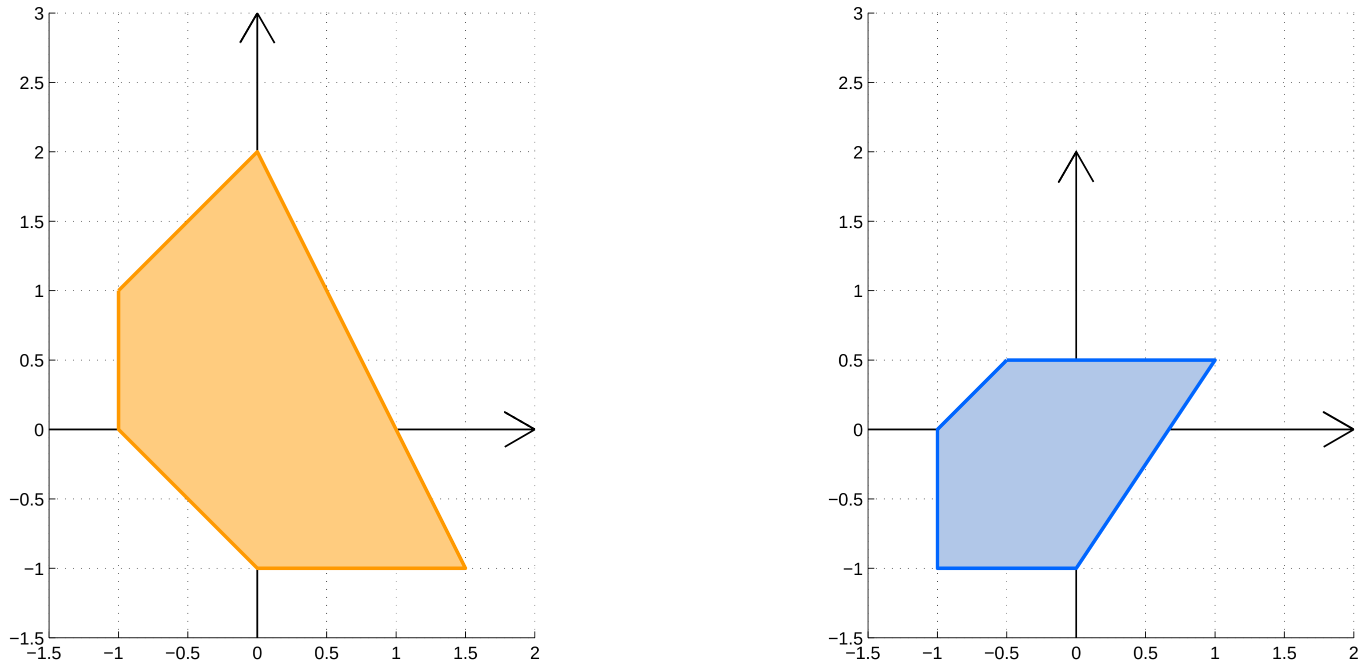
to handle equality constraints, either use $x \geq A^T\lambda$ and $x \leq A^T\lambda$, or use inference rules with unsigned multipliers

polytopes and duality

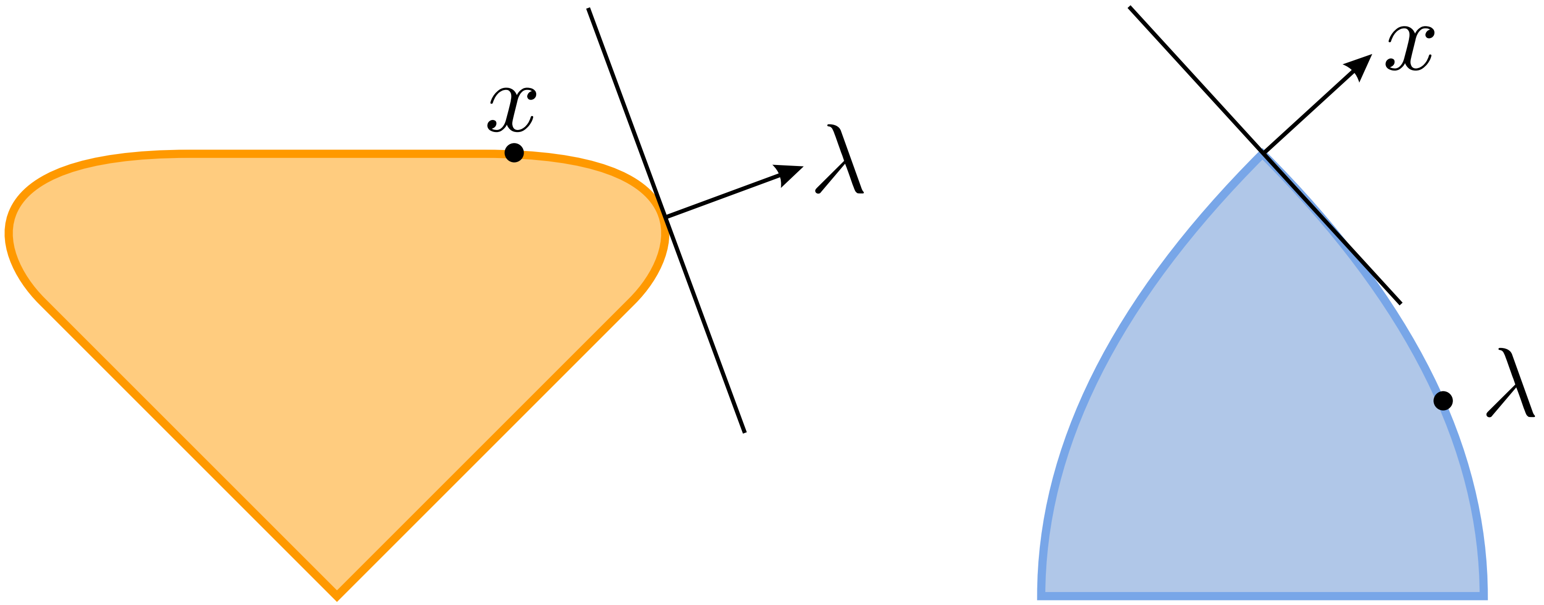
for $S \subset \mathbb{R}^n$ define the *polar set*

$$S^* = \left\{ \lambda \in \mathbb{R}^n \mid \lambda^T x \leq 1 \text{ for all } x \in S \right\}$$

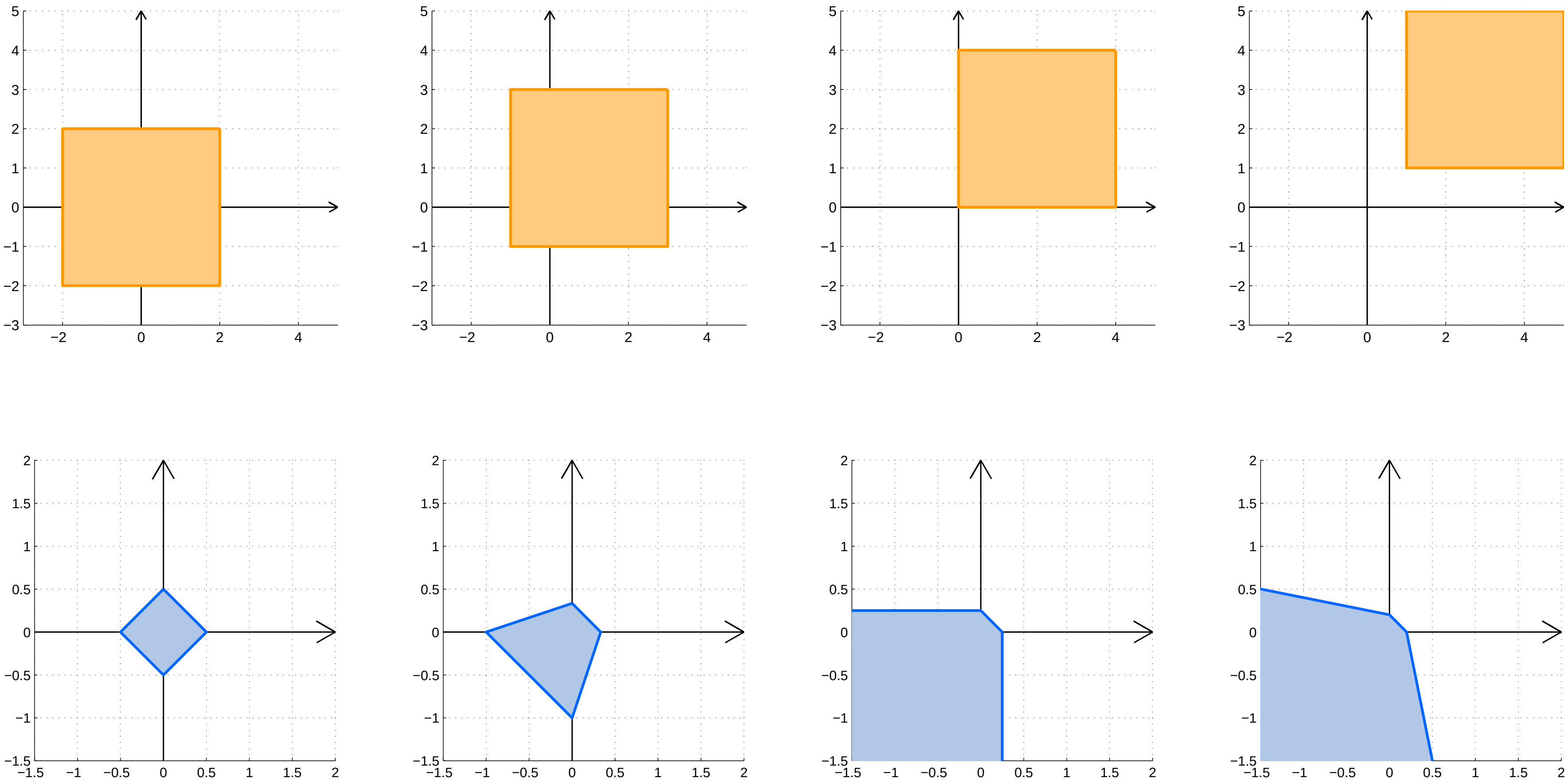
- the polar of a polytope is a polytope
- facets of one correspond to vertices of the other



polar sets



polar sets



properties of polar sets

- the polar S^* depends on the position of S; it is not *affine invariant*
- $0 \in S^*$ for any S
- $P \subset Q$ implies that $P^* \supset Q^*$
- P^* is always convex, even if P is not convex
- if $0 \in P$, then $P = (P^*)^*$

polarity and representations

suppose S is a V-polytope, and $0 \in \mathbf{int}(S)$

$$\begin{aligned} S &= \mathbf{co}\{\, a_1, \ldots, a_m \,\} \subset \mathbb{R}^n \\ &= \{\, A^T\lambda \mid \lambda \geq 0,\ 1^T\lambda = 1 \,\} \end{aligned}$$

then S^* is the H-polytope

$$S^* = \{\, x \mid Ax \leq 1 \,\}$$

- given a polytope S in V-representation, then one also has an H-representation of S^*
- since $S^{**} = S$, if S is the polytope $\{\, x \mid Ax \leq 1 \,\}$ and $0 \in \mathbf{int}(S)$ then $S^* = \mathbf{co}\{\, a_1, \ldots, a_m \,\}$

converting between representations

we can use polarity to convert between representations

given an H-polytope S, we'd like to construct a V-representation

- construct the polar S^*
- it is a V-polytope
- construct the H-representation for S^* using Fourier-Motzkin
- construct $S = S^{**}$, which is a V-polytope, as desired

projection is exponential

the polar of the cube is the *crosspolytope*

$$C_n^* = \mathbf{co}\,\{\, e_1, -e_1, \ldots, e_n, -e_n \,\}$$

with $2n$ vertices and 2^n facets

this is the projection of

$$\Big\{\, (\lambda, x) \;\Big|\; \lambda \geq 0,\ \lambda^T 1 = 1,\ x = A^T \lambda \,\Big\}$$

where the rows of A are $e_1^T, -e_1^T, \ldots, e_n^T, -e_n^T$.

in this case, projecting a polytope defined by $4n+2$ inequalities from $3n$ dimensions to n dimensions results in 2^n facets

computing with representations

we have

$$y \in \mathbb{S}^* \qquad \Longleftrightarrow \qquad S \subset \{\, x \in \mathbb{R}^n \mid y^T x \leq 1 \,\}$$

hence testing membership for S^* is equivalent to testing halfspace containment of S

so we have two problems

- test membership of an H-polytope (or equivalently, test halfspace containment for a V-polytope)
- test membership of a V-polytope (or equivalently, test halfspace containment of an H-polytope)

the first is easy (just evaluation), the second is harder (an LP)

double description

recall Fourier-Motzkin projects an H-polytope onto $x_1 = 0$

i.e., it takes the vectors defining the facets, and constructs new valid inequalities with normal vectors c having $c_1 = 0$

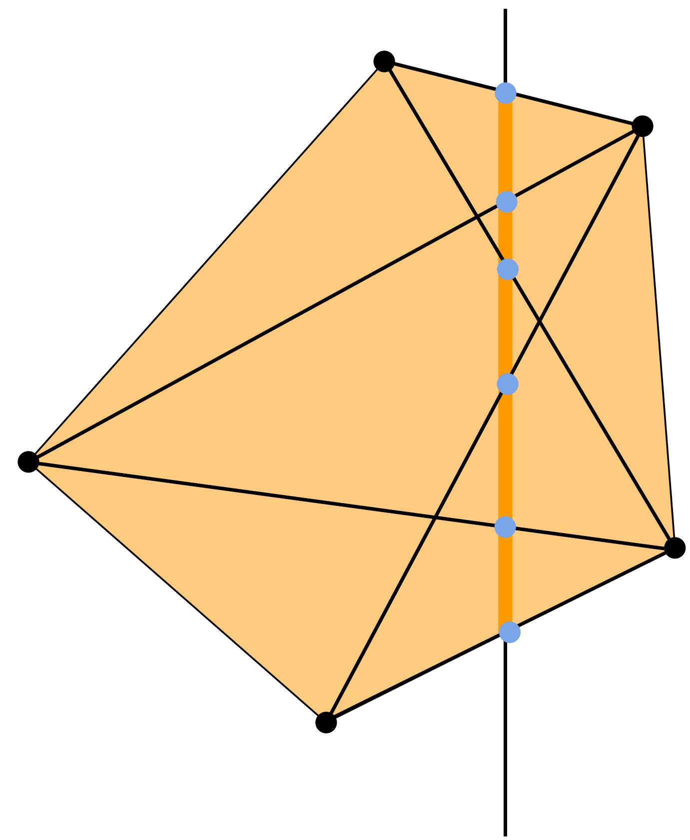
the vectors $a_1, \ldots, a_m$ defining the facets of S also define (after normalization) the vertices of S^*

applying FM gives new vertices c with $c_1 = 0$

one can show that FM constructs the *intersection* of a V-polytope with $x_1 = 0$

this is called the *double description method*

double description



the algorithm is simple:

for each pairs of vertices, one in $x_1 < 0$, the other in $x_1 > 0$, find the intersection with $x_1 = 0$ of the line segment joining them

these new points, together with any points in the original vertex set in $x_1 = 0$, give a V-representation of $S \cap \{x | x_1 = 0\}$

of course, numerically this is the same algorithm as Fourier-Motzkin

polytopes and combinatorial optimization

recall the MAXCUT problem

$$\begin{array}{rl} \text{maximize} & \mathbf{trace}(QX) \\ \text{subject to} & \mathbf{diag}\, X = 1 \\ & \mathbf{rank}(X) = 1 \\ & X \succeq 0 \end{array}$$

the *cut polytope* is the set

$$\begin{aligned} C &= \mathbf{co}\{\, X \in \mathbb{S}^n \mid X = vv^T,\ v \in \{-1,1\}^n \,\} \\ &= \mathbf{co}\{\, X \in \mathbb{S}^n \mid \mathbf{rank}(X) = 1,\ \mathbf{diag}(X) = 1,\ X \succeq 0 \,\} \end{aligned}$$

- maximizing $\mathbf{trace}\, QX$ over $X \in C$ gives exactly the MAXCUT value
- this is equivalent to a *linear program*

MAXCUT

Although we can formulate MAXCUT as an LP, both the V-representation and the H-representation are exponential in the number of vertices

- e.g., for $n = 7$, the cut polytope has $116,764$ facets
  for $n = 8$, there are approx. $217,000,000$ facets

note that this does not necessarily imply that the problem is hard; there are combinatorial problems for which, even though the polytope has an exponential number of facets, there is a polynomial-time *separation oracle*

also several families of valid linear inequalities are known, e.g., the *triangle inequalities* which give LP relaxations of MAXCUT

polytopes for combinatorial problems

there are integer programming formulations of many combinatorial problems

e.g., TSP, 8 nodes gives a 20 dimensional polytope with $194,187$ facets and 2520 vertices

but projecting a polytope dramatically increases the number of facets

the key question: is the cut polytope the projection of some high-dimensional polytope with few facets

if so, then we can replace the original LP with a simpler LP in higher dimensions

this is called the problem of *efficient representation* of MAXCUT; since MAXCUT is NP-complete, such a representation is unlikely to be found