16. More Fourier-Motzkin Elimination

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- Polytopes, MAXCUT, and combinatorial optimization
efficiency of Fourier-Motzkin elimination

if $A$ has $m$ rows, then after elimination of $x_1$ we can have no more than

$$\left\lfloor \frac{m^2}{4} \right\rfloor$$

facets

- if $m/2$ inequalities have a positive coefficient of $x_1$, and $m/2$ have a negative coefficient, then FM constructs exactly $m^2/4$ new inequalities
- repeating this, eliminating $d$ dimensions gives

$$\left\lfloor \frac{m}{2} \right\rfloor^{2n}$$

inequalities

- key question: how may are are redundant? i.e., does projection produce exponentially more facets?
representation of polytopes

we can represent a polytope in the following ways

- *an intersection of halfspaces*, called an $H$-polytope

  $$S = \left\{ x \in \mathbb{R}^n \mid Ax \leq b \right\}$$

- *the convex hull of its vertices*, called a $V$-polytope

  $$S = \text{co}\left\{ a_1, \ldots, a_m \right\}$$
size of representations

in some cases, one representation is smaller than the other

- the $n$-cube

$$C_n = \left\{ x \in \mathbb{R}^n \mid -1 \leq x_i \leq 1 \text{ for all } i \right\}$$

has $2n$ facets, and $2^n$ vertices

- the $n$-dimensional crosspolytope

$$C_n^* = \left\{ x \in \mathbb{R}^n \mid \sum_i |x_i| \leq 1 \right\}$$

$$= \text{co} \{ e_1, -e_1, \ldots, e_n, -e_n \}$$

has $2n$ vertices and $2^n$ facets
optimization problems

the *optimization problem*: given polytope $S$ and $c \in \mathbb{R}^n$, find $x$ that solves

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x \in S
\end{align*}$$

or state that $S = \emptyset$

roughly speaking, an equivalent problem (via bisection search) is *halfspace containment*

given $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, is it true that

$$S \subset \left\{ x \in \mathbb{R}^n \mid c^T x \leq \gamma \right\}$$

if not, find $x \in S$ such that $c^T x > \gamma$
**membership problems**

the *membership problem*: given polytope $S$ and $y \in \mathbb{R}^n$, decide if $y \in S$, and if not find $\lambda \in \mathbb{R}^n$ such that

$$\lambda^T y > \max \{ \lambda^T x \mid x \in S \}$$

the membership problem is also called the *separation problem*
problem solving using different representations

- $V$-polytope: optimization is easy; evaluate $c^T x$ at all vertices for membership, we need to solve an LP; duality will give certificate of infeasibility

- $H$-polytope: membership is easy; evaluate $Ay - b$
  the certificate of infeasibility is just the violated inequality

  the optimization is an LP
converting between representations

suppose we are given a $V$-polytope

$$S = \text{co} \{ a_1, \ldots, a_m \}$$

$$= \left\{ A^T \lambda \mid \lambda \geq 0, \ \lambda^T 1 = 1 \right\}$$

$$= \left\{ x \mid \text{there exists } \lambda \text{ such that } \lambda \geq 0, \ \lambda^T 1 = 1, \ x = A^T \lambda \right\}$$

hence $S$ is a projection onto $\lambda = 0$ of

$$\left\{ (\lambda, x) \mid \lambda \geq 0, \ \lambda^T 1 = 1, \ x = A^T \lambda \right\}$$

so we can use Fourier-Motzkin!

to handle equality constraints, either use $x \geq A^T \lambda$ and $x \leq A^T \lambda$, or use inference rules with unsigned multipliers
polytopes and duality

for $S \subset \mathbb{R}^n$ define the *polar set*

$$S^* = \left\{ \lambda \in \mathbb{R}^n \mid \lambda^T x \leq 1 \text{ for all } x \in S \right\}$$

- the polar of a polytope is a polytope
- facets of one correspond to vertices of the other
polar sets
polar sets

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More Fourier-Motzkin Elimination

S. Lall, Stanford 2003.11.29.01
properties of polar sets

- the polar $S^*$ depends on the position of $S$; it is not affine invariant
- $0 \in S^*$ for any $S$
- $P \subset Q$ implies that $P^* \supset Q^*$
- $P^*$ is always convex, even if $P$ is not convex
- if $0 \in P$, then $P = (P^*)^*$
polarity and representations

suppose $S$ is a $V$-polytope, and $0 \in \text{int}(S)$

$$S = \text{co}\{ a_1, \ldots, a_m \} \subset \mathbb{R}^n$$

$$= \{ A^T \lambda \mid \lambda \geq 0, \ 1^T \lambda = 1 \}$$

then $S^*$ is the $H$-polytope

$$S^* = \{ x \mid Ax \leq 1 \}$$

- given a polytope $S$ in $V$-representation, then one also has an $H$-representation of $S^*$
- since $S^{**} = S$, if $S$ is the polytope $\{ x \mid Ax \leq 1 \}$ and $0 \in \text{int}(S)$ then $S^* = \text{co}\{ a_1, \ldots, a_m \}$
**converting between representations**

we can use polarity to convert between representations

given an $H$-polytope $S$, we’d like to construct a $V$-representation

- construct the polar $S^*$
- it is a $V$-polytope
- construct the $H$-representation for $S^*$ using Fourier-Motzkin
- construct $S = S^{**}$, which is a $V$-polytope, as desired
projection is exponential

the polar of the cube is the crosspolytope

\[ C_n^* = \text{co} \left\{ e_1, -e_1, \ldots, e_n, -e_n \right\} \]

with \(2n\) vertices and \(2^n\) facets

this is the projection of

\[ \left\{ (\lambda, x) \mid \lambda \geq 0, \lambda^T 1 = 1, x = A^T \lambda \right\} \]

where the rows of \(A\) are \(e_1^T, -e_1^T, \ldots, e_n^T, -e_n^T\).

in this case, projecting a polytope defined by \(4n + 2\) inequalities from \(3n\) dimensions to \(n\) dimensions results in \(2^n\) facets
computing with representations

we have

\[ y \in S^* \iff S \subset \{ x \in \mathbb{R}^n \mid y^T x \leq 1 \} \]

hence testing membership for \( S^* \) is equivalent to testing halfspace containment of \( S \)

so we have two problems

- test membership of an \( H \)-polytope (or equivalently, test halfspace containment for a \( V \)-polytope)
- test membership of a \( V \)-polytope (or equivalently, test halfspace containment of an \( H \)-polytope)

the first is easy (just evaluation), the second is harder (an LP)
**double description**

recall Fourier-Motzkin projects an $H$-polytope onto $x_1 = 0$

i.e., it takes the vectors defining the facets, and constructs new valid inequalities with normal vectors $c$ having $c_1 = 0$

the vectors $a_1, \ldots, a_m$ defining the facets of $S$ also define (after normalization) the vertices of $S^*$

applying FM gives new vertices $c$ with $c_1 = 0$

one can show that FM constructs the *intersection* of a $V$-polytope with $x_1 = 0$

this is called the *double description method*
**double description**

the algorithm is simple:

for each pairs of vertices, one in $x_1 < 0$, the other in $x_1 > 0$, find the intersection with $x_1 = 0$ of the line segment joining them

these new points, together with any points in the original vertex set in $x_1 = 0$, give a $V$-representation of $S \cap \{x | x_1 = 0\}$

of course, numerically this is the same algorithm as Fourier-Motzkin
polytopes and combinatorial optimization

recall the MAXCUT problem

\[
\begin{align*}
\text{maximize} & \quad \text{trace}(QX) \\
\text{subject to} & \quad \text{diag} \ X = 1 \\
& \quad \text{rank}(X) = 1 \\
& \quad X \succeq 0
\end{align*}
\]

the cut polytope is the set

\[
C = \text{co}\{ X \in \mathbb{S}^n \mid X = vv^T, \ v \in \{-1, 1\}^n \}
= \text{co}\{ X \in \mathbb{S}^n \mid \text{rank}(X) = 1, \ \text{diag}(X) = 1, \ X \succeq 0 \}
\]

- maximizing \( \text{trace} \ QX \) over \( X \in C \) gives exactly the MAXCUT value
- this is equivalent to a linear program
MAXCUT

Although we can formulate MAXCUT as an LP, both the $V$-representation and the $H$-representation are exponential in the number of vertices

- e.g., for $n = 7$, the cut polytope has 116,764 facets
- for $n = 8$, there are approx. 217,000,000 facets

Note that this does not necessarily imply that the problem is hard; there are combinatorial problems for which, even though the polytope has an exponential number of facets, there is a polynomial-time separation oracle

Also several families of valid linear inequalities are known, e.g., the triangle inequalities which give LP relaxations of MAXCUT
polytopes for combinatorial problems

there are integer programming formulations of many combinatorial problems
e.g., TSP, 8 nodes gives a 20 dimensional polytope with 194,187 facets and 2520 vertices

but projecting a polytope dramatically increases the number of facets

the key question: is the cut polytope the projection of some high-dimensional polytope with few facets

if so, then we can replace the original LP with a simpler LP in higher dimensions

this is called the problem of efficient representation of MAXCUT; since MAXCUT is NP-complete, such a representation is unlikely to be found