

## 13. Semialgebraic Lifting

- Primal and dual formulations so far
- Valid inequalities for the primal
- Lifting
- Primal SDP relaxation
- Positivstellensatz and duality
- Convex relaxation of semialgebraic sets
- The cut polytope
- A general scheme
- Distinguished representations
- Proof lengths

## Primal and Dual Formulations So Far

*Positivity of one polynomial*: does there exist  $x$  such that  $f(x) < 0$ ?

- Dual SDP relaxation:  $f$  is SOS
- Primal SDP relaxation: lifting

*Semialgebraic feasibility*: does there exist  $x$  such that  $f_i(x) \geq 0$  and  $h_j(x) = 0$  for all  $i, j$

- Positivstellensatz is exact dual. Finite degree condition is an SDP: does there exist  $s_i, r_{ij}, t_i$  such that  $s_i, r_{ij}$  is SOS and

$$-1 = s_0 + \sum_i s_i f_i + \sum_{i,j} r_{ij} f_i f_j + \cdots + \sum_i t_i h_i$$

- Questions: what is the dual? It should give a *convex relaxation* of the primal feasible set

## Valid Inequalities for the Primal

Does there exist  $x \in \mathbb{R}^n$  such that

$$f_i(x) \geq 0 \quad \text{for all } i = 1, \dots, m$$

We can add a *parametrized family* of valid inequalities of the form

$$f_i(x)(a_{00} + a_{10}x + a_{01}y + a_{11}xy + \dots)^2 \geq 0$$
$$(a_{00} + a_{10}x + a_{01}y + a_{11}xy + \dots)^2 \geq 0$$

- Any vector  $a$  of coefficients defines a valid inequality
- The multipliers are *squares*; i.e., extreme rays of the SOS cone

The Lagrange duality construction forms linear combinations of these, resulting in a dual with *SOS multipliers*

## Lifting

We can represent these multipliers as

$$a^T z = [a_{00} \ a_{10} \ a_{01} \ a_{11} \ \dots] \begin{bmatrix} 1 \\ x \\ y \\ \vdots \end{bmatrix}$$

so an equivalent feasibility problem is: does there exist  $x$  such that

$$\begin{aligned} f_i(x)(a^T z)^2 &\geq 0 && \text{for all } a, i \\ (a^T z)^2 &\geq 0 && \text{for all } a \end{aligned}$$

now lift; let  $Y = zz^T$ , then we have

$$(a^T z)^2 = a^T Y a$$

## Lifted Problem

The lifted problem is: does there exist  $x \in \mathbb{R}^n$  such that

$$\begin{aligned} a^T (f_i(x)Y) a &\geq 0 && \text{for all } a, i \\ a^T Y a &\geq 0 && \text{for all } a \\ Y &= z z^T \end{aligned}$$

Since  $Y$  defines a quadratic form, we have equivalently

$$\begin{aligned} f_i(x)Y &\succeq 0 && \text{for all } i \\ Y &\succeq 0 \\ Y &= z z^T \end{aligned}$$

## Example

suppose  $f(x) = x^2 + 3x + 1$ ; does there exist  $x$  such that  $f(x) < 0$ ?

Apply the lifting

$$Y = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T = \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix}$$

Then

$$\begin{aligned} f(x)Y &= \begin{bmatrix} x^2 + 3x + 1 & x^3 + 3x^2 + x & x^4 + 3x^3 + x^2 \\ & x^4 + 3x^3 + x^2 & x^5 + 3x^4 + x^3 \\ & & x^6 + 3x^5 + x^4 \end{bmatrix} \\ &= \begin{bmatrix} Y_{13} + 3Y_{12} + Y_{11} & Y_{23} + 3Y_{13} + Y_{12} & Y_{33} + 3Y_{23} + Y_{13} \\ \cdot & \cdot & * \\ \cdot & * & * \\ \cdot & \cdot & \cdot \end{bmatrix} \end{aligned}$$

## Primal SDP Relaxation

Relaxing the constraint  $Y = zz^T$ , we have the SDP

$Y$  is Hankel

$$Y_{11} = 1$$

$$Z = \begin{bmatrix} Y_{13} + 3Y_{12} + Y_{11} & Y_{23} + 3Y_{13} + Y_{12} \\ Y_{23} + 3Y_{13} + Y_{12} & Y_{33} + 3Y_{23} + Y_{13} \end{bmatrix}$$

$$Z \succeq 0$$

$$Y \succeq 0$$

- We have relaxed the valid inequality  $f(x)Y \succeq 0$  to positivity of its principal  $2 \times 2$  submatrix
- We can include as many monomials  $z$  as we like

## SDP Dual

The SDP dual is: does there exist  $\alpha, \lambda, P$  such that

$$\begin{bmatrix} -\alpha & 0 & -\lambda \\ 0 & 2\lambda & 0 \\ -\lambda & 0 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2P_{11} & 3P_{11} + 2P_{12} & P_{11} + 6P_{12} + P_{22} \\ & 0 & 2P_{12} + 3P_{22} \\ & & P_{22} \end{bmatrix} \succeq 0$$

$$P \succeq 0$$

$$\alpha > 0$$

To interpret this, multiply left and right by  $z^T$  and  $z$ , giving

$$-\alpha - (x^2 + 3x + 1)(P_{11} + 2P_{12}x + P_{22}x^2) \text{ is SOS}$$

$$(P_{11} + 2P_{12}x + P_{22}x^2) \text{ is SOS}$$

that is

$$-\alpha = s_0 + s_1 f$$



## Positivstellensatz and Duality

We have the Positivstellensatz refutation

$$-\alpha = s_0 + \sum_i s_i f_i$$

- *Dual SDP relaxation*: express the SOS constraints as SDP constraints
- *Primal SDP relaxation*: relax the *lifting*

$$f_i(x)Y \succeq 0 \quad \text{for all } i$$

$$Y \succeq 0$$

$$Y_{11} = 1$$

$$Y = \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix}^T$$

## Convex Relaxation of Semialgebraic Sets

Given a semialgebraic set, we have the lifting

$$\begin{aligned} f_i(x)Y &\succeq 0 \\ Y &\succeq 0 \\ Y_{11} &= 1 \\ Y &= \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix}^T \end{aligned}$$

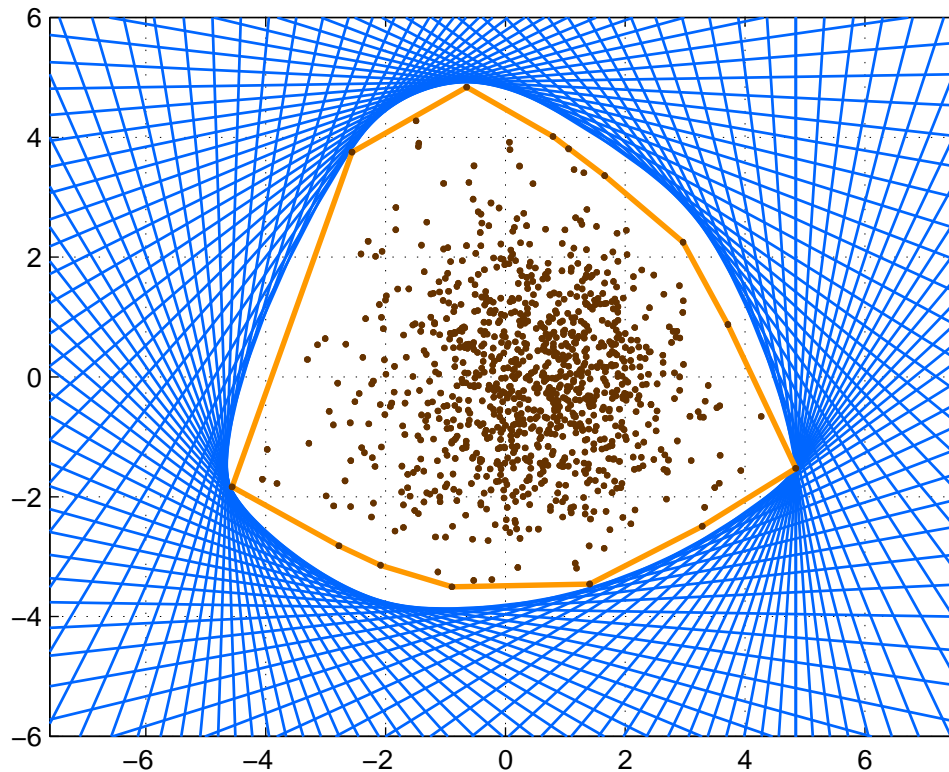
- Projecting the feasible set onto the space spanned by  $x$  gives a convex relaxation of the original semialgebraic set
- We don't need to compute the projection explicitly
- To tighten the relaxation, include more monomials in  $Y$  – equivalently, increase the degree of the multipliers in the refutation

# The Cut Polytope

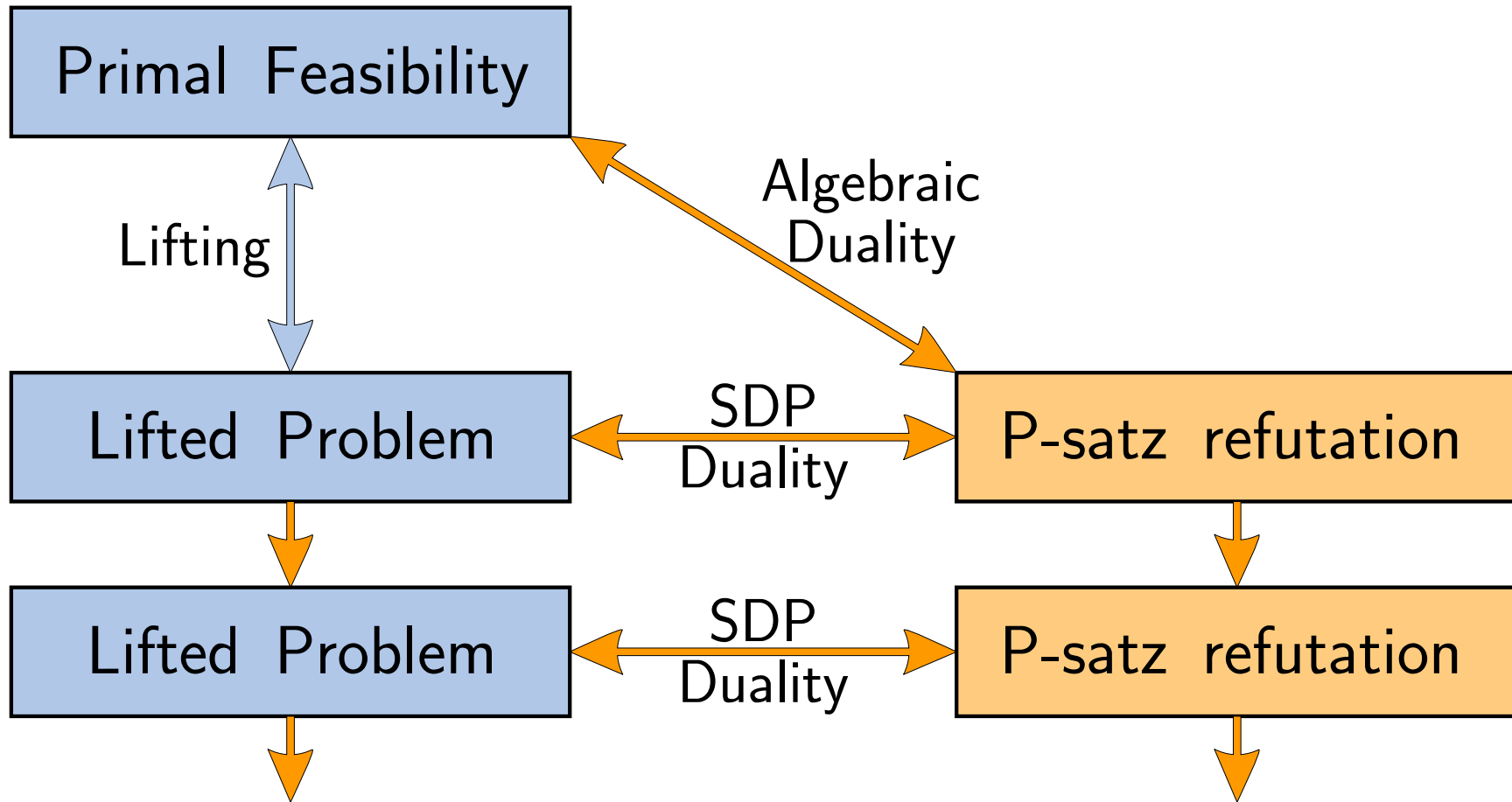
The feasible set of the MAXCUT problem is

$$C = \{ X \in \mathbb{S}^n \mid X = vv^T, v \in \{-1, 1\}^n \}$$

A simple SDP relaxation gives the outer approximation to its convex hull  
Here  $n = 11$ ; the set has affine dimension 55; a projection is shown below



## A General Scheme



## Distinguished Representations

We have a basic semialgebraic  $S$

$$S = \left\{ x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for all } i = 1, \dots, m \right\}$$

Which polynomials are non-negative on  $S$ ?

- Every polynomial in  $\text{cone}\{g_1, \dots, g_m\}$  is non-negative on  $S$
- But are there others? Recall radicality of ideals.

The Positivstellensatz gives an exact test, since  $f(x) \geq 0$  for all  $x \in S$  iff

$$\left\{ x \in \mathbb{R}^n \mid f(x) < 0, g_i(x) \geq 0 \right\} \text{ is empty}$$

## Distinguished Representations

If  $S$  is *compact*, then Schmüdgen showed

$$f(x) > 0 \text{ for all } x \in S \quad \implies \quad f \in \mathbf{cone}\{g_1, \dots, g_m\}$$

- More explicitly, this means

$$f = s_0 + \sum_i s_i g_i + \sum_{i,j} r_{ij} g_i g_j + \dots$$

for some SOS polynomials  $s_i, r_{ij}, \dots$

- Also notice

$$f(x) \geq 0 \text{ for all } x \in S \quad \longleftarrow \quad f \in \mathbf{cone}\{g_1, \dots, g_m\}$$

## Certificate of Positivity

The Positivstellensatz implies  $f(x) \geq 0$  on  $S$  if and only if

$$sf = 1 + s_0 + \sum_i s_i g_i + \sum_{i,j} r_{ij} g_i g_j + \cdots$$

- Schmüdgen's distinguished representation implies that, to prove *strict* positivity, one can assume the multiplier  $s$  is a nonnegative constant
- i.e., one can *prove* positivity using *fewer axioms*. Consequently
  - proofs may become longer
  - need assumptions on  $S$
- So we can *fix* the multiplier  $s$ , without *theoretical* loss, but this may require *higher degree* certificates
- Theoretical justification for optimization of polynomials over compact domains; e.g., Lyapunov stability in a basin of attraction

## Reducing the Axiom Set

If there is a *single* polynomial  $g_k$  such that

$$\left\{ x \in \mathbb{R}^n \mid g_k(x) \geq 0 \right\} \text{ is compact}$$

then Putinar's result holds:

$$f(x) > 0 \text{ for all } x \in S \quad \implies \quad f = s_0 + \sum_i s_i g_i \text{ for some SOS } s_i$$

- Stronger assumptions about  $S$  mean we can reduce axiom set further; we don't need to take products



## Handelman Representations

Suppose that  $S$  is defined by *linear inequalities*

$$S = \left\{ x \in \mathbb{R}^n \mid b - Ax \geq 0 \right\}$$

and  $S$  is compact, with nonempty interior.

Then, if  $f(x) > 0$ , we have for  $W \subset \mathbb{N}^m$

$$f = \sum_{\alpha \in W} c_{\alpha} \prod_{i=1}^m (b_i - a_i^T x)^{\alpha_i} \quad \text{for some } c_{\alpha} > 0$$

- No SOS polynomials, just constants  $c_{\alpha}$ . Hence solvable using LP
- But proofs may be extremely long

## Distinguished Representations

	Products	No products
SOS coefficients	Schmüdgen <i>compactness</i>	Putinar <i>compactness++</i>
Scalar coefficients	Handelman <i>compactness</i> <i>linear inequalities</i>	Lagrange <i>convexity</i> <i>constraint qualifications</i>

- *Strong duality* results
- Positivstellensatz requires no assumptions
- Tradeoffs between computation, assumptions, and proof lengths