14. Sparse Polynomials

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Minkowski sum

for subsets $S, T \subset \mathbb{R}^n$, the *Minkowski sum* is

$$S + T = \left\{ x + y \mid x \in S, \ y \in T \right\}$$

also for $\lambda \in \mathbb{R}$, define

$$\lambda S = \left\{ \lambda x \mid x \in S \right\}$$
convolution

for \( S \in \mathbb{R}^N \) define the \textit{indicator function} \( I_S : \mathbb{R}^N \rightarrow \mathbb{R} \)

\[
I_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{otherwise}
\end{cases}
\]

then the Minkowski sum corresponds to convolution

\[
I_{S+T} = I_S \ast I_T
\]

that is

\[
I_{S+T}(x) = \int_y I_S(x - y)I_T(y) \, dy
\]
properties

if $S$ and $T$ are convex, so is $S + T$

to see this, notice that the Cartesian product is convex

\[
S \times T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \in S, \ y \in T \right\}
\]

and the sum $S + T$ is image of the $S \times T$ under the linear map

\[
\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + y
\]
properties

in general $S + S \neq 2S$, for example

$$S = \{0, 1\} \quad \text{and} \quad S + S = \{0, 1, 2\}$$

if $S$ is convex, then

$$(\lambda + \mu)S = \lambda S + \mu S$$
polyhedra

A set \( S \subset \mathbb{R}^n \) is called a *polyhedron* if it is the intersection of a finite set of closed halfspaces

\[
S = \left\{ x \in \mathbb{R}^n \mid Ax \leq b \right\}
\]

- A bounded polyhedron is called a *polytope*
- The *dimension* of a polyhedron is the dimension of its affine hull
  \[
  \text{affine}(S) = \left\{ \lambda x + \nu y \mid \lambda + \nu = 1, \ x, y \in S \right\}
  \]
- If \( b = 0 \) the polyhedron is a cone
- Every polyhedron is convex
faces of polyhedra

given $a \in \mathbb{R}^n$, the corresponding face of polyhedron $P$ is

$$\text{face}(a, P) = \left\{ x \in P \mid a^T x \geq a^T y \text{ for all } y \in P \right\}$$

- faces of dimension 0 are called vertices
- edges
- facets, where $d = \text{dim}(P)$

- facets are also said to have codimension 1
faces of polyhedra

- if $F$ is a face of $G$, and $G$ is a face of $P$ then $F$ is a face of $P$
  i.e., *is a face of* is transitive

- $\text{face}(a, S + T) = \text{face}(a, S) + \text{face}(a, T)$

- in particular, if $x$ is a vertex of $S + T$, then

  $$x = y + z$$

  for some $y$, a vertex of $S$ and $z$, a vertex of $T$

  and the vertices $y$ and $z$ are unique
positive polynomials

suppose \( f = c_dx^d + c_{d-1}x^{d-1} + \cdots + c_1x + c_0 \); then

\[ f \text{ is PSD} \quad \implies \quad d \text{ is even, } c_d > 0 \text{ and } c_0 \geq 0 \]

what is the analogue in \( n \) variables?
example

- suppose \( f = x^3y^2 + xy + 1 \)

substitute \( x = t \) and \( y = t \), i.e., evaluate \( f \) along the curve \( x = y \),

\[ \hat{f} = t^5 + t^2 + 1 \]

so clearly \( f \) is not PSD

this suggests that \( f \) is PSD implies \( f \) has even degree

- but for \( f = x^3y^2 - xy^4 + x^2y^2 + 1 \) the same substitution gives

\[ \hat{f} = t^4 + 1 \]
the Newton polytope

suppose

\[ f = \sum_{\alpha \in M} c_\alpha x^\alpha \]

the set of monomials \( M \subset \mathbb{N}^n \) is called the \textit{frame} of \( f \)

the \textit{Newton polytope} of \( f \) is its convex hull

\[ \text{new}(f) = \text{co}(\text{frame}(f)) \]

the example shows

\[ f = 7 x^4 y + x^3 y + x^2 y^4 + x^2 + 3 x y \]
necessary condition for nonnegativity

we’ll evaluate the polynomial \( f \) along the curve

\[
x_1 = z_1 t^{a_1} \\
\vdots \\
x_n = z_n t^{a_n}
\]

for \( f = \sum_{\alpha \in M} c_\alpha x^\alpha \) define

\[
\hat{f} = \sum_{\alpha \in M} c_\alpha z^\alpha t^{a^T \alpha}
\]

e.g., for \( f = x^3 y + 2xy^7 \) we have

\[
\hat{f} = z_1^3 z_2 t^{3a_1 + a_2} + 2 z_1 z_2^7 t^{a_1 + 7a_2}
\]
necessary condition for nonnegativity

if \( f \in \mathbb{R}[x_1, \ldots, x_n] \) is PSD, then

every vertex of \( \text{new}(f) \) has even coordinates, and a positive coefficient

- \( f = 7x^4y + x^3y + x^2y^4 + x^2 + 3xy \)
  is not PSD, since term \( 3xy \) has coords \((1, 1)\)

- \( f = 7x^4y + x^3y - x^2y^4 + x^2 + 3y^2 \)
  is not PSD, since term \(-x^2y^4\) has a negative coefficient
**proof**

if $\beta$ is a vertex of $\text{new}(f)$, then there is some $a \in \mathbb{R}^n$ such that

$$a^T \beta > a^T \alpha \text{ for all } \alpha \in M$$

evaluating $\hat{f}$ along the curve $x_i = z_i t^{a_i}$, gives

$$\hat{f} = c_\beta z^\beta t^{a^T \beta} + \text{ terms of lower degree in } t$$
as $t \to \infty$, the first terms dominates, so

$$c_\beta z^\beta \geq 0 \text{ for all } z \in \mathbb{R}^n$$

assume $f$ is PSD, then

- picking $z = 1$ implies $c_\beta$ must be positive
- picking $z_j = -1$ and $z_i = 1$ for $i \neq j$ implies $\beta_i$ must be even
halfspaces containing the Newton polytope

the Newton polytope of $f$ is contained with the halfspace specified by $a, b$

$$\text{new}(f) \subset \left\{ x \in \mathbb{R}^n \mid a^T x \leq b \right\}$$

if and only if

$$\lim_{t \to \infty} |t^{-b} \hat{f}| < \infty \quad \text{for all } z \in \mathbb{R}^n$$
example

suppose \( a = [1 \ 1]^T \) and \( b = 6 \)

\[
f = -x^4 y^2 + x^2 y^4 + x y + 1
\]

then

\[
\hat{f} = (-z_1^4 z_2^2 + z_1^2 z_2^4) t^6 + z_1 z_2 t^2 + 1
\]

we have

\[
\text{new}(f) \subset \left\{ x \in \mathbb{R}^n \mid a^T x \leq b \right\} \quad \Longrightarrow \quad \lim_{t \to \infty} |t^{-b} \hat{f}| < \infty
\]

the converse also holds, since by picking \( z \) arbitrarily we can arrange for the leading coefficient of \( \hat{f} \) to be non-zero
Newton polytopes of a product

\[ \text{new}(fg) = \text{new}(f) + \text{new}(g) \]

\[
\begin{align*}
  f &= x^4 y^2 + 2 x^2 y^3 - x^2 - x y^2 \\
  g &= x^3 - y + 1 \\
  fg &= x^7 y^2 + 2 x^5 y^3 - x^5 - x^4 y^3 - 2 x^2 y^4 + 2 x^2 y^3 \\
      &\quad + x^2 y - x^2 + x y^3 - x y^2
\end{align*}
\]
Newton polytopes

we’d like to show \( \text{new}(fg) = \text{new}(f) + \text{new}(g) \)

first, we’ll show

\[ \text{new}(fg) \subset \text{new}(f) + \text{new}(g) \]

to see this, if \( f = \sum_{\alpha} c_{\alpha} c_{\alpha} \) and \( g = \sum_{\beta} d_{\beta} x^{\beta} \) then

\[ fg = \sum_{\alpha} \sum_{\beta} c_{\alpha} d_{\beta} x^{\alpha + \beta} \]

so \( \text{frame}(fg) \subset \text{frame}(f) + \text{frame}(g) \)

also we have \( \text{co}(S + T) \subset \text{co}(S) + \text{co}(T) \)
Newton polytopes

it remains to show

\[ \text{new}(fg) \supset \text{new}(f) + \text{new}(g) \]

we’ll show that if \( \gamma \) is a vertex of \( \text{new}(f) + \text{new}(g) \) then \( \gamma \in \text{new}(fg) \)

we know \( \gamma = \alpha + \beta \) for unique \( \alpha \in \text{frame}(f) \) and \( \beta \in \text{frame}(g) \)

\( \alpha \) and \( \beta \) are unique since \( \gamma \) is a vertex

the coefficient of \( x^\gamma \) in \( fg \) is \( c_\alpha d_\beta \), which cannot be zero, so \( \gamma \in \text{new}(fg) \)
Newton polytopes of squares

consequently we have

\[
\text{new}(f^n) = n \text{ new}(f)
\]

with

\[
f = x^4 y^2 + 2 x^2 y^3 - x^2 - x y^2
\]

we have

\[
f^2 = x^8 y^4 + 4 x^6 y^5 - 2 x^6 y^2 - 2 x^5 y^4
\]
\[
+ 4 x^4 y^6 - 4 x^4 y^3
\]
\[
+ x^4 - 4 x^3 y^5 + 2 x^3 y^2 + x^2 y^4
\]
Newton polytopes and inequalities

if \( f \) and \( g \) are PSD polynomials then

\[
\begin{align*}
\text{if } f(x) \leq g(x) \text{ for all } x \in \mathbb{R}^n & \implies \text{new}(f) \subset \text{new}(g) \\
\text{we’ll show that any halfspace containing } \text{new}(g) \text{ also contains } \text{new}(f) \\
\text{if } \text{new}(g) \subset \{ x \mid a^T x \leq b \} \text{ then }
\end{align*}
\]

\[
\lim_{t \to \infty} t^{-b} \hat{g} < \infty \quad \text{for all } z
\]

since \( 0 \leq f \leq g \) we therefore have the same holds for \( \hat{f} \), and so

\[
\text{new}(f) \subset \{ x \mid a^T x \leq b \}
\]
example

\[ f = x^4 y^2 + 2 x^2 y^3 - x^2 - x y^2 \]

new\( (f^2) \)

new\( (f^2(x^2 y^2 + x^4 + 1)) \)
sparse SOS decomposition

this tells us which monomials we have in an SOS decomposition

\[
f = \sum_{i=1}^{t} g_i^2 \implies \text{new}(g_i) \subseteq \frac{1}{2} \text{new}(f)
\]

because \( 0 \leq g_i^2 \leq f \) so

\[
\text{new}(f) \supseteq \text{new}(g_i^2) = 2 \text{new}(g_i)
\]

this holds for every SOS decomposition of \( f \)
example: sparse SOS decomposition

find an SOS representation for

\[ f = 4x^4y^6 + x^2 - xy^2 + y^2 \]

the squares in an SOS decomposition can only contain the monomials

\[ \text{new}(\frac{1}{2} f) \cap \mathbb{N}^n = \{ x^2y^3, xy^2, xy, x, y \} \]

without using sparsity, we would include all 21 monomials of degree < 5 in the SDP

with sparsity, we only need 5 monomials
example continued

we find

\[ f = 4x^4y^6 + x^2 - xy^2 + y^2 \]

\[
f = \begin{bmatrix}
y \\
x \\
xy \\
xy^2 \\
x^2y^3
\end{bmatrix}^T \begin{bmatrix}
1 & 0 & -0.5 & 0 & -0.5 \\
0 & 1 & 0 & -0.5 & 0 \\
-0.5 & 0 & 1 & 0 & 0 \\
0 & -0.5 & 0 & 1 & 0 \\
-0.5 & 0 & 0 & 0 & 4
\end{bmatrix} \begin{bmatrix}
y \\
x \\
xy \\
xy^2 \\
x^2y^3
\end{bmatrix}
\]

and the matrix is PSD
homogeneous polynomials

A polynomial $f$ is called homogeneous if

$$f = \sum_{\alpha \in M} c_{\alpha} x^{\alpha} \quad \text{with} \quad \sum_{i=1}^{n} \alpha_i = d \text{ for all } \alpha \in M$$

If $f$ is homogeneous, then for an SOS decomposition we need only look at monomials $x^\beta$ such that

$$\sum_{i=1}^{n} \beta_i = \frac{d}{2}$$

For example

$$f = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

$$= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$
software

- SOSTOOLS – www.cds.caltech.edu/sostools
- GloptiPoly – www.laas.fr/~henrion/software/gloptipoly