

10. Sum of Squares

- Polynomial nonnegativity
- Sum of squares (SOS) decomposition
- Example of SOS decomposition
- Computing SOS using semidefinite programming
- Convexity
- Positivity in one variable
- Background
- Global optimization
- Optimizing in parameter space
- Lyapunov functions

polynomial programming

so far

- polynomial equations over the complex field

objectives

- general quantified formulae
- with Boolean connectives
- polynomial equations, inequalities, and inequations over the reals

e.g., does there exist x such that for all y

$$(f(x, y) \geq 0) \wedge (g(x, y) = 0) \vee (h(x, y) \neq 0)$$

polynomial nonnegativity

first, consider the case of one inequality; given $f \in \mathbb{R}[x_1, \dots, x_n]$

does there exist $x \in \mathbb{R}^n$ such that $f(x) < 0$

- if not, then f is globally non-negative

$$f(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

and f is called *positive semidefinite* or *PSD*

- the problem is *NP-hard*, but decidable
- many applications

certificates

the problem

does there exist $x \in \mathbb{R}^n$ such that $f(x) < 0$

- answer yes is easy to verify; exhibit x such that $f(x) < 0$
- answer no is hard; we need a *certificate* or a *witness*
i.e, a proof that there is no feasible point

Sum of Squares Decomposition

if there are polynomials $g_1, \dots, g_s \in \mathbb{R}[x_1, \dots, x_n]$ such that

$$f(x) = \sum_{i=1}^s g_i^2(x)$$

then f is nonnegative

an easily checkable certificate, called a *sum-of-squares (SOS)* decomposition

- how do we find the g_i
- when does such a certificate exist?

example

we can write any polynomial as a *quadratic function of monomials*

$$\begin{aligned}
 f &= 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \\
 &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
 &= z^T Q(\lambda) z
 \end{aligned}$$

which holds for all $\lambda \in \mathbb{R}$

if for some λ we have $Q(\lambda) \succeq 0$, then we can factorize $Q(\lambda)$

example, continued

e.g., with $\lambda = 6$, we have

$$Q(\lambda) = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

so

$$\begin{aligned} f &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\ &= \left\| \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy - 3y^2 \end{bmatrix} \right\|^2 \\ &= (2x^2y^2 + y^2)^2 + (2x^2 + xy - 3y^2)^2 \end{aligned}$$

which is an SOS decomposition

sum of squares and semidefinite programming

suppose $f \in \mathbb{R}[x_1, \dots, x_n]$, of degree $2d$

let z be a vector of all monomials of degree less than or equal to d

f is SOS if and only if there exists Q such that

$$\begin{array}{l} Q \succeq 0 \\ f = z^T Q z \end{array}$$

- this is an SDP in standard primal form
- the number of components of z is $\binom{n+d}{d}$
- comparing terms gives affine constraints on the elements of Q

sum of squares and semidefinite programming

if Q is a feasible point of the SDP, then to construct the SOS representation

factorize $Q = VV^T$, and write $V = [v_1 \dots v_r]$, so that

$$\begin{aligned} f &= z^T VV^T z \\ &= \|V^T z\|^2 \\ &= \sum_{i=1}^r (v_i^T z)^2 \end{aligned}$$

- one can factorize using e.g., Cholesky or eigenvalue decomposition
- the number of squares r equals the rank of Q

example

$$\begin{aligned}
 f &= 2x^4 + 2x^3y - x^2y^2 + 5y^4 \\
 &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
 &= q_{11}x^4 + 2q_{12}x^3y + (q_{22} + 2q_{13})x^2y^2 + 2q_{23}xy^3 + q_{33}y^4
 \end{aligned}$$

so f is SOS if and only if there exists Q satisfying the SDP

$$\begin{aligned}
 Q \succeq 0 & & q_{11} &= 2 & 2q_{12} &= 2 \\
 & & 2q_{12} + q_{22} &= -1 & 2q_{23} &= 0 \\
 & & q_{33} &= 5 & &
 \end{aligned}$$

convexity

the sets of PSD and SOS polynomials are a *convex cones*; i.e.,

$$f, g \text{ PSD} \quad \implies \quad \lambda f + \mu g \text{ is PSD for all } \lambda, \mu \geq 0$$

let $P_{n,d}$ be the set of PSD polynomials of degree $\leq d$

let $\Sigma_{n,d}$ be the set of SOS polynomials of degree $\leq d$

- both $P_{n,d}$ and $\Sigma_{n,d}$ are *convex cones* in \mathbb{R}^N where $N = \binom{n+d}{d}$
- we know $\Sigma_{n,d} \subset P_{n,d}$, and testing if $f \in P_{n,d}$ is NP-hard
- but testing if $f \in \Sigma_{n,d}$ is an SDP (but a large one)

polynomials in one variable

if $f \in \mathbb{R}[x]$, then f is SOS if and only if f is PSD

example

all real roots must have even multiplicity, and highest coeff. is positive

$$\begin{aligned} f &= x^6 - 10x^5 + 51x^4 - 166x^3 + 342x^2 - 400x + 200 \\ &= (x - 2)^2 (x - (2 + i)) (x - (2 - i)) (x - (1 + 3i)) (x - (1 - 3i)) \end{aligned}$$

now reorder complex conjugate roots

$$\begin{aligned} &= (x - 2)^2 (x - (2 + i)) (x - (1 + 3i)) (x - (2 - i)) (x - (1 - 3i)) \\ &= (x - 2)^2 ((x^2 - 3x - 1) - i(4x - 7)) ((x^2 - 3x - 1) + i(4x - 7)) \\ &= (x - 2)^2 ((x^2 - 3x - 1)^2 + (4x - 7)^2) \end{aligned}$$

so every PSD scalar polynomial is the sum of *one or two* squares

quadratic polynomials

a quadratic polynomial in n variables is PSD if and only if it is SOS

because it is PSD if and only if

$$f = x^T Q x$$

where $Q \geq 0$

and it is SOS if and only if

$$\begin{aligned} f &= \sum_i (v_i^T x)^2 \\ &= x^T \left(\sum_i v_i v_i^T \right) x \end{aligned}$$

some background

In 1888, Hilbert showed that PSD=SOS if and only if

- $d = 2$, i.e., quadratic polynomials
- $n = 1$, i.e., univariate polynomials
- $d = 4, n = 2$, i.e., quartic polynomials in two variables

$n \backslash d$	2	4	6	8
1	yes	yes	yes	yes
2	yes	yes	no	no
3	yes	no	no	no
4	yes	no	no	no

- in general f is PSD does not imply f is SOS

some background

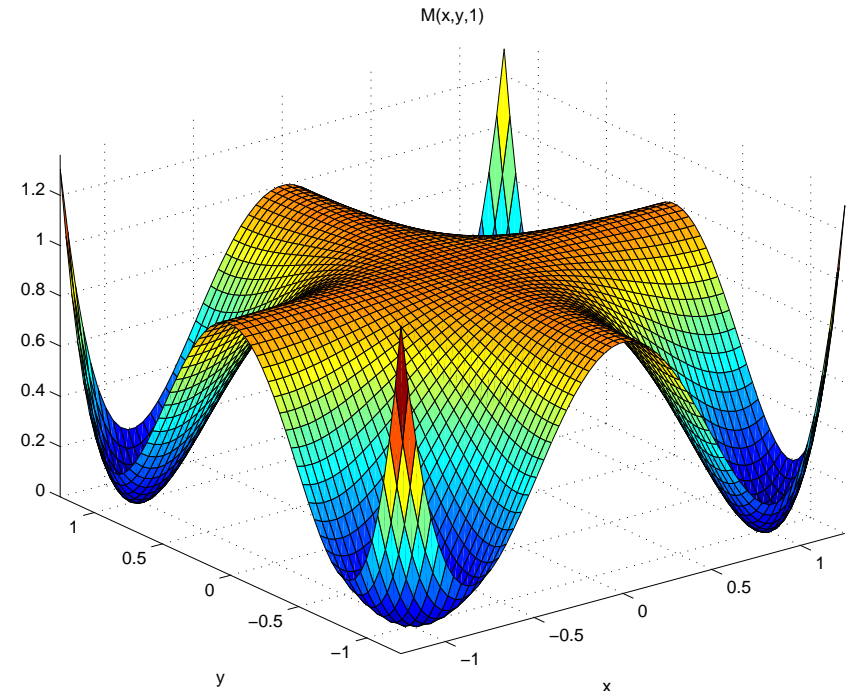
- Connections with Hilbert's 17th problem, solved by Artin: every PSD polynomial is a SOS of *rational functions*.
- If f is not SOS, then can try with gf , for some g .
 - For fixed f , can optimize over g too
 - Otherwise, can use a “universal” construction of Pólya-Reznick.

More about this later.

The Motzkin Polynomial

A positive semidefinite polynomial, that is *not* a sum of squares.

$$M(x, y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$



- Nonnegativity follows from the arithmetic-geometric inequality applied to $(x^2y^4, x^4y^2, 1)$
- Introduce a nonnegative factor $x^2 + y^2 + 1$
- Solving the SDPs we obtain the decomposition:

$$\begin{aligned} (x^2 + y^2 + 1) M(x, y) = & (x^2y - y)^2 + (xy^2 - x)^2 + (x^2y^2 - 1)^2 + \\ & + \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xy)^2 \end{aligned}$$

The Univariate Case:

$$\begin{aligned}
 f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2d}x^{2d} \\
 &= \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}^T \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0d} \\ q_{01} & q_{11} & \cdots & q_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ q_{0d} & q_{1d} & \cdots & q_{dd} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix} \\
 &= \sum_{i=0}^d \left(\sum_{j+k=i} q_{jk} \right) x^i
 \end{aligned}$$

- In the univariate case, the SOS condition is exactly equivalent to non-negativity.
- The matrices A_i in the SDP have a Hankel structure. This can be exploited for efficient computation.

About SOS/SDP

- The resulting SDP problem is polynomially sized (in n , for fixed d).
- By properly choosing the monomials, we can exploit structure (sparsity, symmetries, ideal structure).
- An important feature: the problem is still a SDP *if the coefficients of F are variable*, and the dependence is affine.
- Can optimize over SOS polynomials in affinely described families.

For instance, if we have $p(x) = p_0(x) + \alpha p_1(x) + \beta p_2(x)$, we can “easily” find values of α, β for which $p(x)$ is SOS.

This fact will be *crucial* in everything that follows...

Global Optimization

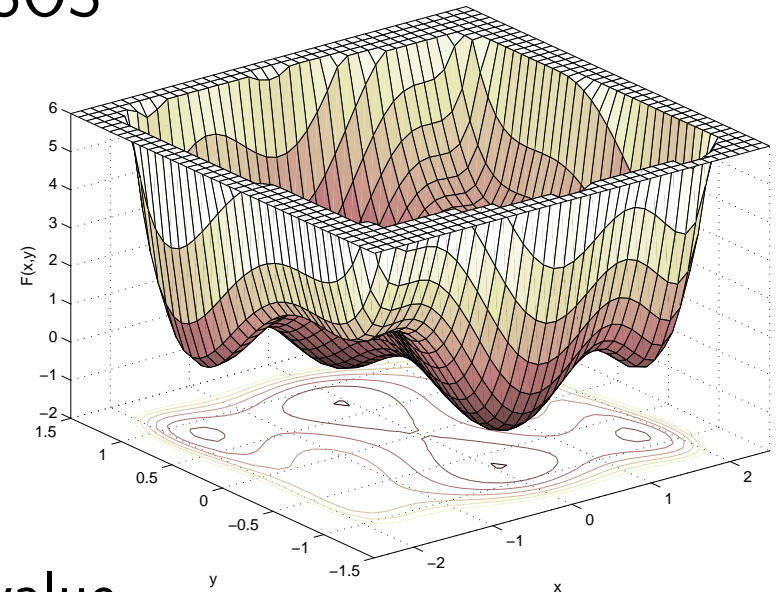
Consider the problem

$$\min_{x,y} f(x, y)$$

with

$$f(x, y) := 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4$$

- Not convex. Many local minima. NP-hard.
- Find the largest γ s.t. $f(x, y) - \gamma$ is SOS
- Essentially due to Shor (1987).
- A semidefinite program (convex!).
- If exact, can recover optimal solution.
- *Surprisingly* effective.



Solving, the maximum γ is -1.0316. Exact value.

Coefficient Space

Let $f_{\alpha\beta}(x) = x^4 + (\alpha + 3\beta)x^3 + 2\beta x^2 - \alpha x + 1$.

What is the set of values of $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha\beta}$ is PSD? SOS?

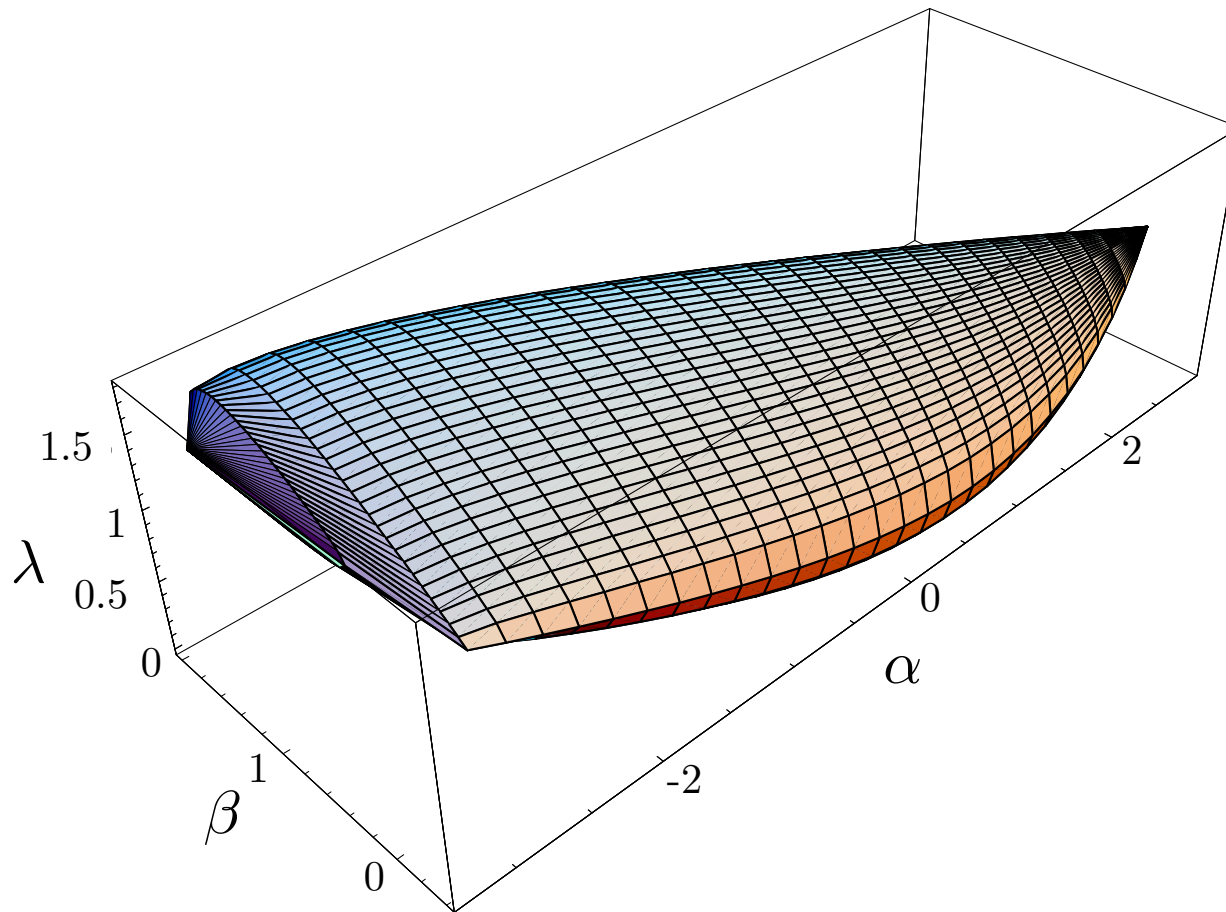
To find a SOS decomposition:

$$\begin{aligned}
 f_{\alpha,\beta}(x) &= 1 - \alpha x + 2\beta x^2 + (\alpha + 3\beta)x^3 + x^4 \\
 &= \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\
 &= q_{11} + 2q_{12}x + (q_{22} + 2q_{13})x^2 + 2q_{23}x^3 + q_{33}x^4
 \end{aligned}$$

The matrix Q should be PSD and satisfy the affine constraints.

The feasible set is given by:

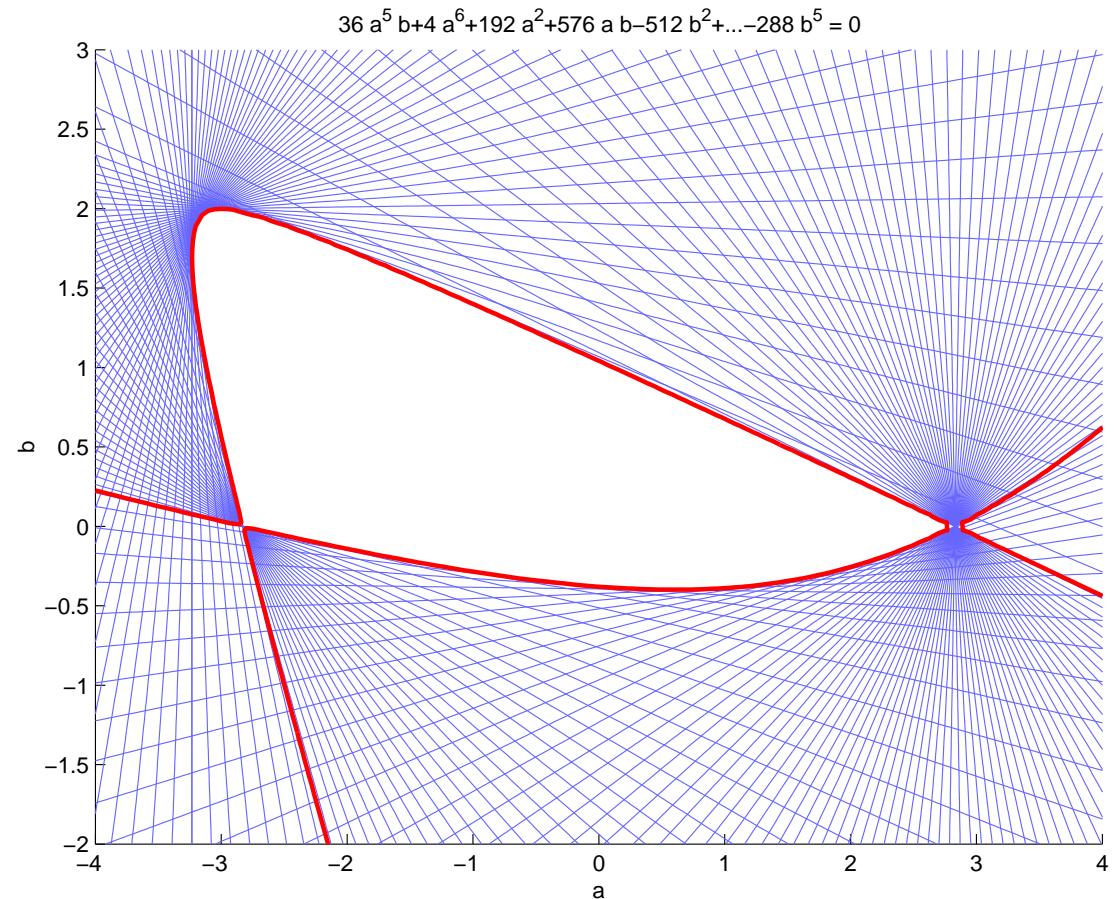
$$\left\{ (\alpha, \beta) \mid \exists \lambda \text{ s.t. } \begin{bmatrix} 1 & -\frac{1}{2}\alpha & \beta - \lambda \\ -\frac{1}{2}\alpha & 2\lambda & \frac{1}{2}(\alpha + 3\beta) \\ \beta - \lambda & \frac{1}{2}(\alpha + 3\beta) & 1 \end{bmatrix} \succeq 0 \right\}$$



What is the set of values of $(\alpha, \beta) \in \mathbb{R}^2$ for which $f_{\alpha\beta}$ PSD? SOS?

Recall: in the univariate case PSD=SOS, so here the sets are the same.

- Convex and semialgebraic.
- It is the projection of a spectrahedron in \mathbb{R}^3 .
- We can easily test membership, or even optimize over it!



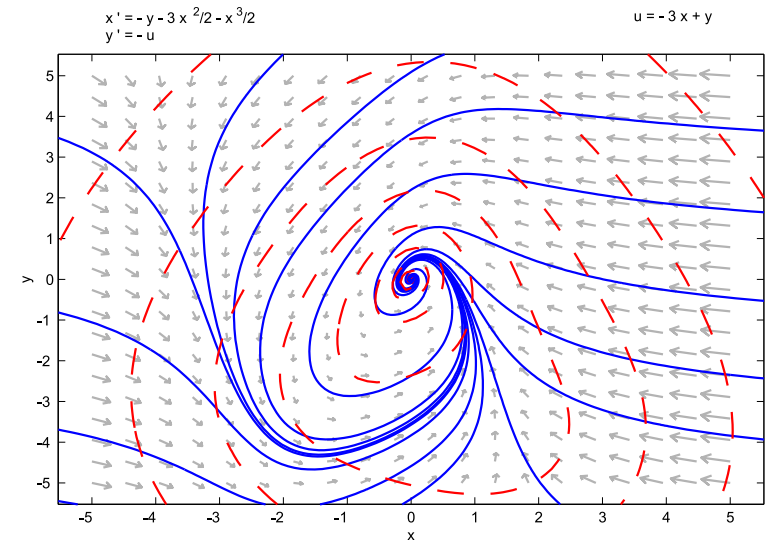
Defined by the curve: $288\beta^5 - 36\alpha^2\beta^4 + 1164\alpha\beta^4 + 1931\beta^4 - 132\alpha^3\beta^3 + 1036\alpha^2\beta^3 + 1956\alpha\beta^3 - 2592\beta^3 - 112\alpha^4\beta^2 + 432\alpha^3\beta^2 + 1192\alpha^2\beta^2 - 1728\alpha\beta^2 + 512\beta^2 - 36\alpha^5\beta + 72\alpha^4\beta + 360\alpha^3\beta - 576\alpha^2\beta - 576\alpha\beta - 4\alpha^6 + 60\alpha^4 - 192\alpha^2 - 256 = 0$

Lyapunov Stability Analysis

To prove asymptotic stability of $\dot{x} = f(x)$,

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x} \right)^T f(x) < 0, \quad x \neq 0$$

$$V(x) > 0 \quad x \neq 0$$



- For linear systems $\dot{x} = Ax$, quadratic Lyapunov functions $V(x) = x^T P x$

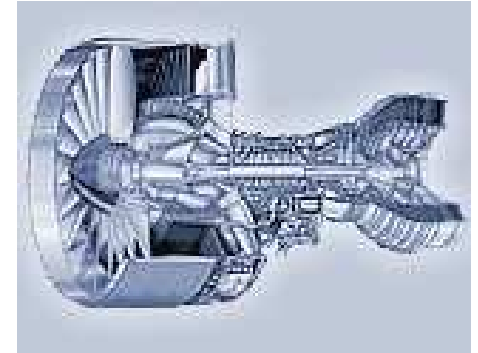
$$P > 0, \quad A^T P + P A < 0.$$

- With an affine family of candidate polynomial V , \dot{V} is also affine.
- Instead of *checking nonnegativity*, use a *SOS condition*.
- Therefore, for polynomial vector fields and Lyapunov functions, we can check the conditions using the theory described before.

Lyapunov Example

A jet engine model (derived from Moore-Greitzer),
with controller:

$$\begin{aligned}\dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\ \dot{y} &= 3x - y\end{aligned}$$



Try a generic 4th order polynomial Lyapunov function.

$$V(x, y) = \sum_{0 \leq j+k \leq 4} c_{jk} x^j y^k$$

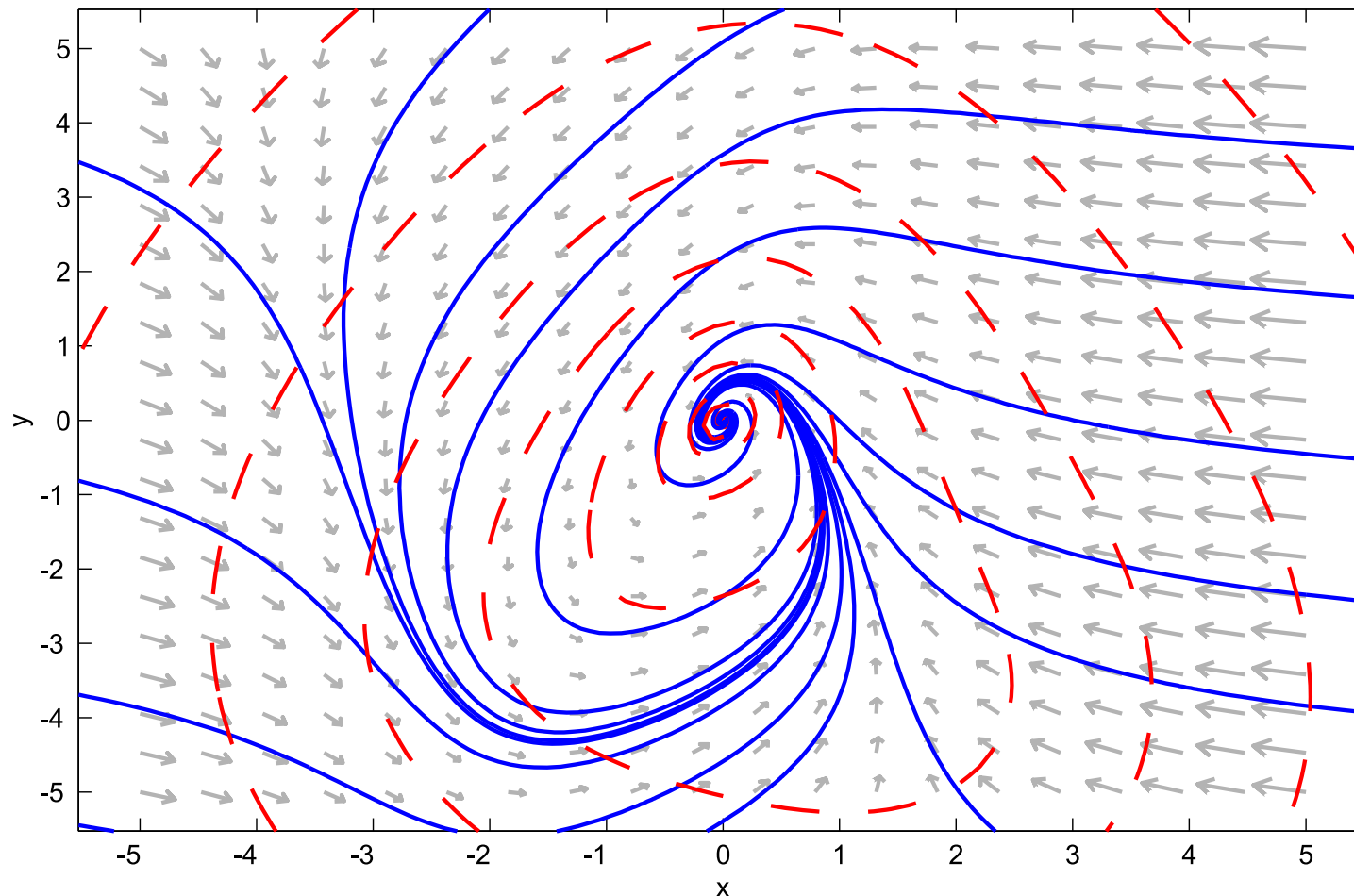
Find a $V(x, y)$ that satisfies the conditions:

- $V(x, y)$ is SOS.
- $-\dot{V}(x, y)$ is SOS.

Both conditions are affine in the c_{jk} . Can do this directly using SOS/SDP!

Lyapunov Example

After solving the SDPs, we obtain a Lyapunov function.



$$V = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 + 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.0000018868xy^3 + 0.090723y^4$$

Lyapunov Example

Find a Lyapunov function for

$$\dot{x} = -x + (1 + x)y$$

$$\dot{y} = -(1 + x)x.$$

we easily find a quartic polynomial

$$V(x, y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4.$$

Both $V(x, y)$ and $(-\dot{V}(x, y))$ are SOS:

$$V(x, y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}, \quad -\dot{V}(x, y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix} 10 & 1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 1 & 12 & 0 \\ 1 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}$$

The matrices are positive definite, so this proves asymptotic stability.

Extensions

- Other linear differential inequalities (e.g. Hamilton-Jacobi).
- Many possible variations: nonlinear \mathcal{H}_∞ analysis, parameter dependent Lyapunov functions, etc.
- Can also do local results (for instance, on compact domains).
- Polynomial and rational vector fields, or functions with an underlying algebraic structure.
- Natural extension of the LMIs for the linear case.