A New Approach for Analysis and Synthesis of Time-Varying Systems

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***Abstract*— In this paper new techniques are developed for the analysis of linear time-varying (LTV) systems. These lead to a formally simple treatment of robust control problems for LTV systems, allowing methods more usually restricted to time-invariant systems to be employed in the time-varying case. As an illustration of this methodology, the so-called H_∞ synthesis problem is solved for LTV systems.**

***Index Terms*— LMI, periodic systems, robust control, time variation.**

I. Introduction

In this paper, new techniques are developed for the analysis of linear time-varying (LTV) systems. These lead to a formally simple treatment of problems for LTV systems, allowing methods usually restricted to time-invariant systems to be employed in the time-varying case. Analysis and synthesis techniques for LTV systems can be applied to control of nonlinear systems along trajectories and for design of multirate filters in signal processing.

We make use of the fact that the usual state-space description of an LTV system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned}$$

described by time-varying matrices A_k, B_k, C_k, and D_k, is equivalent to a description in terms of *block-diagonal* operators. This leads to an operator-based description of the system and a function which takes the role of a transfer function for time-varying systems.

We show that this function, called the *system function*, has many properties analogous to those of transfer functions of linear time-invariant (LTI) systems. In particular, for LTI systems, the induced norm is the maximum of a matrix norm over frequency, and in the time-varying case a very similar result is true. This allows us to apply techniques which have formerly been restricted to LTI systems to LTV systems. In doing this, many of the proofs become *formally* identical, and this leads to extremely simple derivations. In particular, and most importantly, this makes the machinery and results of robust control available for LTV systems.

Manuscript received April 11, 1997; revised June 30, 1998. Recommended by Associate Editor, A. Rantzer.

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Publisher Item Identifier S 0018-9286(99)06242-X.

We apply these techniques to the H_∞ analysis and synthesis problem for LTV systems. In the spirit of recent results on LTI systems using linear matrix inequalities (LMI's) [9], [16], we derive a solution for the LTV case expressed in terms of *linear operator* inequalities. The derivation is formally identical to that used in the LTI case. The method also gives some insight into the nature of the relationship between the Riccati equation and LMI solutions and their connection to particular structured singular value problems. The papers [3] and [11] consider similar synthesis problems using a closely related approach; a key distinction of the current paper is its generality and connections with standard robust control techniques and the compact derivation of the results and machinery.

The techniques presented here also render simple the solution of the H_∞ synthesis problem for periodically time-varying discrete systems. The periodicity of the system leads naturally to a solution expressed in terms of finite-dimensional linear matrix inequalities, solvable by standard means.

II. Preliminaries

We now introduce our notation and gather some elementary facts. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The open and closed unit discs of $\mathbb{C}$ are represented by $\mathbb{D}$ and $\bar{\mathbb{D}}$, and $\mathbb{T}$ is the unit circle.

Given a Hilbert space E we denote its norm by $\|\cdot\|_E$ and its inner product by $\langle\cdot,\cdot\rangle_E$; for convenience we frequently suppress the subscript. Given two Hilbert spaces E and F we denote the space of bounded linear operators mapping E to F by $\mathcal{L}(E,F)$ and shorten this to $\mathcal{L}(E)$ when E equals F. If X is in $\mathcal{L}(E,F)$ we denote the E to F induced norm of X by $\|X\|_{E\to F}$. The adjoint of X is written as X^*. When X is in $\mathcal{L}(E)$ we denote its spectrum by $\operatorname{spec}(X)$ which is defined by

$$\operatorname{spec}(X) = \{\lambda \in \mathbb{C} : I\lambda - X \text{ is not invertible in } \mathcal{L}(E)\}.$$

The spectral radius of X will be denoted by $\operatorname{rad}(X)$.

When an operator $X \in \mathcal{L}(E)$ is self-adjoint we use $X < 0$ to mean it is negative definite; that is there exists a number $\alpha > 0$, such that for all nonzero $x \in E$ the inequality

$$\langle x, Xx\rangle < -\alpha\|x\|^2$$

holds. We now state an elementary fact used in the sequel.

Proposition 1: Suppose X and Y are self-adjoint operators on two Hilbert spaces and W is an operator between these spaces. Then

$$\begin{pmatrix} X & W \\ W^* & Y \end{pmatrix} < 0$$

if and only if $Y < 0$ and $X - WY^{-1}W^* < 0$.

This is the well-known Schur complement formula and will be referred to as such; it can be found in any introductory text on matrix or operator theory.

In the sequel we will require the weak operator topology of $\mathcal{L}(F,E)$. A sequence Y_k in $\mathcal{L}(F,E)$ converges weakly to $Y \in \mathcal{L}(F,E)$, denoted $\lim_{k\to\infty} Y_k \overset{\text{weak}}{=} Y$, if for every $y \in F, x \in E$ the following limit holds:

$$\lim_{k\to\infty} \langle x, Y_k y\rangle = \langle x, Yy\rangle.$$

See, for example, [10] for details.

The main Hilbert space of interest in the paper is denoted by $\ell_2(E)$ where E is a Euclidean space. It consists of elements $x = (x_0, x_1, x_2, \cdots)$, with each $x_k \in E$, which satisfy

$$\sum_{k=0}^{\infty} \|x_k\|_E^2 < \infty.$$

The inner product of x, y in $\ell_2(E)$ is therefore defined by the infinite sum $\langle x,y\rangle_{\ell_2} = \sum_{k=0}^{\infty}\langle x_k, y_k\rangle_E$. If the space E is clear from the context we abbreviate $\ell_2(E)$ to ℓ_2.

One of the most important operators used in the paper is the unilateral shift operator Z, defined on $\ell_2(E)$ as follows. For $a = (a_0, a_1, a_2, \cdots)$ in $\ell_2(E)$ we define Za by

$$(Za) = (0, a_0, a_1, a_2, \cdots).$$

We now introduce a more specialized notation for the purposes of this paper.

A. Block-Diagonal Operators

Definition 2: A bounded operator Q mapping $\ell_2(X)$ to $\ell_2(Y)$ is *block-diagonal* if there exists a sequence of operators Q_k in $\mathcal{L}(X,Y)$ such that, for all w, z, if $z = Qw$ then $z_k = Q_k w_k$. Then Q has the representation

$$\begin{bmatrix} Q_0 & & & 0 \\ & Q_1 & & \\ & & Q_2 & \\ 0 & & & \ddots \end{bmatrix}.$$

Further, if $P_k \in \mathcal{L}(X,Y)$ is a uniformly bounded sequence of operators we say $P = \operatorname{diag}(P_0, P_1, \cdots)$ is the block-diagonal operator for P_k, and conversely given P a block-diagonal operator, the blocks are denoted by P_k, for $k \geq 0$.

Suppose F, G, R, and S are block-diagonal operators, and let A be a *partitioned operator*, each of whose elements is a block-diagonal operator, such as

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix}.$$

We now define the following notation:

$$\left[\!\!\left[\begin{matrix} F & G \\ R & S \end{matrix}\right]\!\!\right] := \operatorname{diag}\left(\begin{bmatrix} F_0 & G_0 \\ R_0 & S_0 \end{bmatrix}, \begin{bmatrix} F_1 & G_1 \\ R_1 & S_1 \end{bmatrix}, \cdots\right)$$

which we call the *diagonal realization* of A. Implicit in the definition of $[\![A]\!]$ is the underlying block structure of the partitioned operator A. Clearly, for any given operator A of this particular structure, $[\![A]\!]$ is simply A with the rows and columns permuted appropriately so that

$$\left[\!\!\left[\begin{matrix} F & G \\ R & S \end{matrix}\right]\!\!\right]_k = \begin{bmatrix} F_k & G_k \\ R_k & S_k \end{bmatrix}.$$

Hence there exist permutation operators, which we shall denote by $P_l(A)$ and $P_r(A)$, such that $P_l(A)AP_r(A) = [[A]]$ or equivalently

$$P_l\left(\begin{bmatrix} F & G \\ R & S \end{bmatrix}\right)\begin{bmatrix} F & G \\ R & S \end{bmatrix} P_r\left(\begin{bmatrix} F & G \\ R & S \end{bmatrix}\right) = \left[\!\!\left[\begin{matrix} F & G \\ R & S \end{matrix}\right]\!\!\right].$$

For any operator A whose elements are block-diagonal operators

$$\begin{aligned} P_l(A)P_l(A)^* &= P_l(A)^*P_l(A) = I \\ P_r(A)P_r(A)^* &= P_r(A)^*P_r(A) = I \end{aligned}$$

and if A is self-adjoint, then $P_l(A) = P_r(A)^*$. For a concrete example, consider $[F \quad G]$. Then

$$P_l([F \quad G]) = I, \qquad P_r([F \quad G]) = \begin{bmatrix} E \\ Z^*EZ \end{bmatrix}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & \\ \vdots & & & & & & \ddots \end{bmatrix}.$$

The following is immediate.

Proposition 3: For any real number β, and any partitioned operator A consisting of elements which are block-diagonal, $A < \beta I$ holds if and only if $[\![A]\!] < \beta I$. That is, positivity is preserved under permutation.

Two further useful facts for the above permutations are the following.

Proposition 4:

1) Suppose that A and B are partitioned operators consisting of block-diagonal elements and that their structures are the same. Then

$$[\![A + B]\!] = [\![A]\!] + [\![B]\!].$$

2) Suppose that A and C are partitioned operators, each of which consists of elements which are block-diagonal. Further suppose that the block structures are compatible, so that the product $\hat{A}\hat{C}$ is block-diagonal for any operators $\hat{A}$ and $\hat{C}$ with the same block structures as A and C. Then

$$[\![AC]\!] = [\![A]\!][\![C]\!].$$

Proof: Part 1) is obvious.

Part 2) is simple to see, since $P_r(A)$, the right permutation of A, depends only on the column dimensions of the blocks in A. Since A and C have compatible block structure, $P_l(C) = P_r(A)^*$, and hence

$$\begin{aligned} [\![A]\!][\![C]\!] &= P_l(A)AP_r(A)P_l(C)CP_r(C) \\ &= P_l(A)ACP_r(C) \\ &= P_l(AC)ACP_r(AC) \\ &= [\![AC]\!] \end{aligned}$$

which is the required result. □

III. Linear Time-Varying Systems

We consider a fundamental class of LTV systems in discrete time. The standard way of describing such a system G is using state-space notation

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k \\ z_k &= C_k x_k + D_k w_k \end{aligned} \tag{1}$$

for $w \in \ell_2$, where $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times n_w}$, $C_k \in \mathbb{R}^{n_z\times n}$ and $D_k \in \mathbb{R}^{n_z\times n_w}$ are bounded matrices. The initial condition of the system is $x_0 = 0$.

Our main objective is to develop an operator-based description of such systems. We show that many of the standard state-space methods used in the analysis of LTI systems can be applied directly to time-varying systems using these methods. As an example, we will solve the H_∞ synthesis problem for LTV systems.

Using the previously defined notation, clearly A_k, B_k, C_k, and D_k in (1) define block-diagonal operators. Recalling that Z is the shift, we can rewrite (1) as

$$\begin{aligned} x &= ZAx + ZBw \\ z &= Cx + Dw. \end{aligned}$$

The question of whether this set of equations is well-defined, that is whether or not there exists an $x \in \ell_2$ such that they are satisfied, is one of *stability* of the system. If the equations are well-defined, then we can write

$$G = C(I - ZA)^{-1}ZB + D \tag{2}$$

and $z = Gw$. These equations are clearly well-defined if $1 \notin \operatorname{spec}(ZA)$. The next result shows that this condition is equivalent to the standard notion of stability of LTV systems, that is exponential stability.

Definition 5: The system G is *exponentially stable* if, when $w = 0$, there exist constants $c > 0$ and $0 < \lambda < 1$ such that, for each $k_0 \geq 0$ and any initial condition $x_{k_0} \in \mathbb{R}^n$, the inequality $\|x_k\|_{\mathbb{R}^n} \leq c\lambda^{(k-k_0)}\|x_{k_0}\|_{\mathbb{R}^n}$ holds for all $k \geq k_0$.

Proposition 6: Suppose A_k is a bounded sequence in $\mathcal{L}(X)$ where X is a Hilbert space. Then the difference equation $x_{k+1} = A_k x_k$ is exponentially stable if and only if $1 \notin \operatorname{spec}(ZA)$.

This is the well-known result that exponential stability is equivalent to ℓ_2 stability of the system $x_{k+1} = A_k x_k + v_k$; versions of this result can be found in any standard reference on Lyapunov theory, for instance [19]. Thus the system is stable if and only if $1 \notin \operatorname{spec}(ZA)$; we will work with this latter condition.

Throughout the sequel we will refer to the block-diagonal operators A, B, C, and D and the operator G they define without formal reference to their definitions in (1) and (2).

IV. The System Function

We now consider the properties of operators of the form of (2). Formally, this equation looks very much like the frequency domain description of a discrete-time time-invariant system. It is well known that for such systems, one can replace the shift operator Z with a complex number, and then the induced norm of the system is given by the maximum norm of this transfer function over the unit ball in the complex plane.

We will show that, for linear *time-varying* systems, very similar statements can be made. Indeed, the induced norm of an LTV system can be analyzed by computing the maximum norm of an operator-valued function over a complex ball. However, in this context we will use a bounded sequence $\lambda_k \in \mathbb{C}$ of complex numbers as our notion of frequency. Robust control techniques to date have been primarily developed for LTI systems; the system function derived here provides an important and direct link between LTI and LTV systems, making the techniques of robust control available for LTV systems. In particular, this allows the construction of convex upper bounds for structured uncertainty problems for LTV systems.

Given such a sequence, we will make use of two associated block-diagonal operators. These are

$$\begin{aligned} \Lambda &= \begin{bmatrix} \lambda_0 I & & & 0 \\ & \lambda_1 I & & \\ & & \lambda_2 I & \\ 0 & & & \ddots \end{bmatrix} \\ \Omega &= \begin{bmatrix} \lambda_0 I & & & 0 \\ & \lambda_0\lambda_1 I & & \\ & & \lambda_0\lambda_1\lambda_2 I & \\ 0 & & & \ddots \end{bmatrix} \end{aligned} \tag{3}$$

on ℓ_2. Observe that

$$\Omega Z = \Lambda Z \Omega \tag{4}$$

which is easily verified. Also note that if each element of the sequence λ_k is on the unit circle $\mathbb{T}$ then Ω is invertible in $\mathcal{L}(\ell_2)$. Using the definition of Λ we define the *system function* of the operator G by

$$\hat{G}(\Lambda) := C(I - \Lambda ZA)^{-1}\Lambda ZB + D$$

when the inverse is defined. We can now state the main result of this section.

Theorem 7: Suppose $1 \notin \operatorname{spec}(ZA)$. Then

$$\|C(I - ZA)^{-1}ZB + D\| = \sup_{\lambda_k \in \bar{\mathbb{D}}} \|\hat{G}(\Lambda)\|$$

where Λ depends on λ_k as in (3).

This theorem says that the induced ℓ_2 norm of the system G, which equals $\|C(I - ZA)^{-1}ZB + D\|$, is given by the maximum of the norm $\|\hat{G}(\Lambda)\|$, when the λ_k are chosen in the unit disk. This result looks similar to the well-known result for transfer functions of time-invariant systems, and it is the key element in allowing time-invariant techniques to be applied to time-varying systems.

In particular, we will see that we can use this result to derive a time-varying version of the Kalman–Yacubovitch–Popov (KYP) lemma, characterizing those systems which are contractive. This allows the development of time-varying analogs of well-known results in structured singular value analysis or so-called μ-analysis. However, first we must prove a preliminary result.

Lemma 8: Suppose $1 \notin \operatorname{spec}(ZA)$. Then given any sequence λ_k in $\mathbb{T}$, the operator $I - \Lambda ZA$ is invertible and we have

$$\|C(I-ZA)^{-1}ZB + D\| = \|\hat{G}(\Lambda)\|.$$

Proof: Fix a sequence $\lambda_k \in \mathbb{T}$ and define the operator Ω as in (3). Now notice that both Ω and Ω^{-1} are isometries and therefore

$$\|C(I-ZA)^{-1}ZB+D\| = \|\Omega\{C(I-ZA)^{-1}ZB+D\}\Omega^{-1}\|.$$

To complete the proof consider the operator on the right-hand side above

$$\begin{aligned}\Omega\{C(I-ZA)^{-1}ZB+D\}\Omega^{-1} &= C\Omega(I-ZA)^{-1}Z\Omega^{-1}B+D\\ &= C\Omega(I-\Lambda ZA)^{-1}\Omega^{-1}\Lambda ZB+D\\ &= C(I-\Lambda ZA)^{-1}\Lambda ZB+D = \hat{G}(\Lambda)\end{aligned}$$

where we have used the fact that Ω commutes with A, B, C, and D, and the relationship described by (4). □

This lemma states that it is possible to scale the system matrices A and B by any complex sequence on the unit circle without affecting the norm of the system. Note that this can equivalently be thought of as scaling Z, the shift operator. The next lemma describes the effect of the operator Λ on the spectrum of ZA.

Lemma 9: Suppose that λ_k is a sequence in $\bar{\mathbb{D}}$ and define Λ as in (3).

1) If $\mu \notin \operatorname{spec}(ZA)$, then $\mu \notin \operatorname{spec}(\Lambda ZA)$.
2) If the sequence λ_k is further restricted to be in $\mathbb{T}$, then $\operatorname{spec}(ZA) = \operatorname{spec}(\Lambda ZA)$.

Proof: First note that without loss of generality we may assume that $\mu = 1$ in 1) and therefore will show that $1 \notin \operatorname{spec}(ZA)$ implies that $1 \notin \operatorname{spec}(\Lambda ZA)$.

We begin proving 1) by invoking Proposition 6 to see that, since $1 \notin \operatorname{spec}(ZA)$, the difference equation $x_{k+1} = A_k x_k$ is exponentially stable. Each λ_k satisfies $|\lambda_k| \le 1$ and so

$$x_{k+1} = \lambda_{k+1} A_k x_k$$

is also exponentially stable. Again use Proposition 6 to conclude that $1 \notin \operatorname{spec}(ZQ)$ where Q is the block-diagonal operator corresponding to $Q_k = \lambda_{k+1} A_k$. It is routine to verify that $ZQ = \Lambda ZA$.

Part 2) is immediate by applying (4) to see that $\Omega ZA \Omega^{-1} = \Lambda ZA$. □

Note that in particular 1) and 2) imply, the (apparently) well-known result, that the spectrum of ZA is an entire disc centered at zero[1]; to see this, set $\Lambda = \lambda I$ and let λ be in $\bar{\mathbb{D}}$. We can now prove the main result of this section.

Proof of Theorem 7: For convenience define $\gamma := \|\hat{G}(I)\|$ which is equal to $\|G\|$ by definition. Suppose contrary to the theorem that there exists a sequence $\lambda_k \in \bar{\mathbb{D}}$ such that $\|\hat{G}(\Lambda)\| > \gamma$. Then there exist elements $x, y \in \ell_2$ satisfying $\|x_2\| = \|y\|_2 = 1$ and

$$|\langle y, \hat{G}(\Lambda) w\rangle_2| > \gamma.$$

[1] Operators of the form ZA are commonly known as weighted shifts.

Without loss of generality we may assume that w and y have *finite* support, which we denote by n.

Now it is routine to verify that $\hat{G}(\Lambda)$ is lower triangular and has the representation

$$\hat{G}(\Lambda) = \begin{bmatrix} D_0 & & & 0\\ \lambda_1 T_{10} & D_1 & & \\ \lambda_2\lambda_1 T_{20} & \lambda_2 T_{21} & D_2 & \\ \lambda_3\lambda_2\lambda_1 T_{30} & \vdots & & \ddots \\ \vdots & & & \end{bmatrix} \tag{5}$$

where $T_{kl} = C_k A_{k-1} \cdots A_{l+1} B_l$. Therefore, recalling that w and y have finite support, the inner product

$$\langle y, \hat{G}(\Lambda) w\rangle_2 = p(\lambda_1, \cdots, \lambda_n)$$

where $p(\cdot, \cdots, \cdot)$ is some multinomial. Multinomials satisfy a maximum principle (see for instance [17]); specifically p satisfies

$$\max_{\mu_k \in \bar{\mathbb{D}}} |p(\mu_1, \cdots, \mu_n)| = \max_{\mu_k \in \mathbb{T}} |p(\mu_1, \cdots, \mu_n)|.$$

Thus there exist numbers $\lambda_1', \cdots, \lambda_n'$ on the unit circle $\mathbb{T}$ so that

$$|p(\lambda_1', \cdots, \lambda_n')| \ge |p(\lambda_1, \cdots, \lambda_n)| > \gamma. \tag{6}$$

Let Λ' be the operator, of form (3), that corresponds to the sequence $\{1, \lambda_1', \cdots, \lambda_n', 1, \cdots\}$. Observe that by Lemma 8 we have $\|\hat{G}(I)\| = \|\hat{G}(\Lambda')\|$. Also note that $\hat{G}(\Lambda')$ has the same lower triangular form as $\hat{G}(\Lambda)$ in (5) and therefore

$$\langle y, \hat{G}(\Lambda') w\rangle_2 = p(\lambda_1', \cdots, \lambda_n').$$

Thus by (6) the inequality $|\langle y, \hat{G}(\Lambda') w|_2\rangle > \gamma$ holds.

Now certainly $\|\hat{G}(\Lambda')\| \ge |\langle y, \hat{G}(\Lambda') w\rangle_2|$ and hence $\|\hat{G}(\Lambda')\| > \gamma$; also recall that $\|\hat{G}(I)\| = \|\hat{G}(\Lambda')\|$. But this is a contradiction since by definition $\gamma = \|\hat{G}(I)\|$. □

In the sequel we primarily work with the system function when $\Lambda = \lambda I$, where λ is a complex scalar. Observe by defining the notation

$$\hat{G}(\lambda) := C(I - \lambda ZA)^{-1} \lambda ZB + D$$

this specialized function $\hat{G}(\lambda)$ looks and acts very much like the transfer function of an LTI system and therefore plays an instrumental role in our viewpoint in the next section.

V. Evaluating the ℓ_2-Induced Norm

The previous section showed that the induced norm of an LTV system was given by the maximum of an operator norm over a complex ball. In this section, our primary goal is to show that this can be recast into a convex condition on the system matrices. We will see that the results derived appear very similar to those derived for time-invariant systems, and indeed the methodology parallels that for time-invariant systems.

To start we state the following technical lemma.

Lemma 10: The following conditions are equivalent.

1) $\sup_{\lambda \in \bar{\mathbb{D}}} \|C(I - \lambda ZA)^{-1}\lambda ZB + D\| < 1$ and $\text{rad}(ZA) < 1$.
2) There exists $\bar{X} \in \mathcal{L}(\ell_2)$, which is self-adjoint and $\bar{X} > 0$, such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{7}$$

This is an operator version of a well-known matrix result. It does *not* depend on the structure of A, B, C, or D, or the presence of the operator Z. A proof of this result, which we omit, can be found in [21].

For comparison, the corresponding standard result for LTI systems can be stated as follows; given a system G with transfer function $G(z) := C_0(I - zA_0)^{-1}zB_0 + D_0$ in a minimal realization, the H_∞ norm of G is less than one if and only if there exists a matrix $X_0 > 0$ such that

$$\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}^* \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} - \begin{bmatrix} X_0 & 0 \\ 0 & I \end{bmatrix} < 0.$$

Essentially this matrix result can be stated in many different ways, e.g., in terms of the eigenvalues of a Hamiltonian matrix or in terms of the existence of solutions to a Riccati equation [15]. However, one of the powerful features of the above formulation is that it is *affine* in the variable X_0. This leads to both powerful analytical results and simple computations.

In Lemma 10 the variable $\bar{X}$ has no particular structure except that it is self-adjoint and positive definite and is therefore not directly useful in the current context. Our next goal is therefore to improve upon this and obtain a formulation in which the variable is block-diagonal. To this end define the set $\mathcal{X}$ to consist of positive definite self-adjoint operators X of the form

$$X = \begin{bmatrix} X_0 & & & 0 \\ & X_1 & & \\ & & X_2 & \\ 0 & & & \ddots \end{bmatrix} > 0 \tag{8}$$

where the block structure is the same as that of the operator A. With this definition we can state the main result of this section.

Theorem 11: The following conditions are equivalent.

1) $\|C(I - ZA)^{-1}ZB + D\| < 1$ and $1 \notin \text{spec}(ZA)$.
2) There exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0. \tag{9}$$

Formally, the result is the same as that for the LTI case, but the operators ZA and ZB replace the usual A-matrix and B-matrix, and X is block-diagonal. We shall see in the sequel that this is a general property of this formalism and that this gives a simple way to construct and to understand the relationship between time-invariant and time-varying systems.

Proof: We start by invoking Theorem 7 and Lemma 9 with $\Lambda := \lambda I$: condition 1) above is equivalent to condition 1) in Lemma 10. Therefore, it suffices to show that 2) above is equivalent to 2) in Lemma 10. Also, a solution $X \in \mathcal{X}$ to (9) immediately satisfies 2) in Lemma 10 with $\bar{X} := X$.

It only remains to show that a solution $\bar{X}$ to (7) implies that there exists $X \in \mathcal{X}$ satisfying (9), which we now demonstrate. Suppose $\bar{X} \in \mathcal{L}(\ell_2)$ is self-adjoint and satisfies both $\bar{X} > 0$ and (9). Our goal is to construct $X \in \mathcal{X}$ from $\bar{X}$ and show that it has the desired properties.

Define the operator $E_k = [\underbrace{0 \ \cdots \ 0}_{k \text{ zeros}} I \ 0 \ \cdots]^*$, for $k \geq 0$, mapping $\mathbb{R}^n \to \ell_2$ which then satisfies

$$E_k^* A = [0 \quad \cdots \quad 0 \quad A_k \quad 0 \quad \cdots].$$

Observe that $E_k^* E_k = I$. Using $\bar{E}_k$, define X to be the block-diagonal operator $\ell_2 \to \ell_2$ corresponding to the sequence defined by

$$X_k = E_k^* \bar{X} E_k, \qquad \text{for each } k \geq 0.$$

Thus, X is a block-diagonal operator, whose elements are the blocks on the diagonal of $\bar{X}$. Clearly, X is self-adjoint and satisfies $X > 0$ because $\bar{X}$ has these properties. This proves $X \in \mathcal{X}$.

To complete the proof we must now demonstrate that X satisfies (9). Grouping Z in (9) with X we apply Proposition 3 to see that (9) holds if and only if the permuted inequality

$$\left[\!\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} Z^*XZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right]\!\right] < 0$$

holds. Now we can apply Proposition 4 to show that the above is tantamount to

$$\left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\right]^* \left[\!\left[\begin{matrix} Z^*XZ & 0 \\ 0 & I \end{matrix} \right]\!\right] \left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\right] - \left[\!\left[\begin{matrix} X & 0 \\ 0 & I \end{matrix} \right]\!\right] < 0. \tag{10}$$

We will now show that this inequality is satisfied.

Observe that, for each $k \geq 0$, the following holds[2]:

$$E_k^* C = [0 \quad \cdots \quad 0 \quad C_k \quad 0 \quad \cdots].$$

Now using the facts $E_k^* E_k = I$, it is routine to verify the important property

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & E_k \end{bmatrix} = \begin{bmatrix} E_k & 0 \\ 0 & E_k \end{bmatrix} \left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\right]_k \quad \text{holds} \tag{11}$$

for each $k \geq 0$.

Since $\bar{X}$ by assumption satisfies (7) there exists a $\beta > 0$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} Z^*\bar{X}Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} < -\beta I.$$

[2] Here we do not distinguish between versions of E_k that differ only in the spatial dimension of the identity block.

Pre- and postmultiply this by $\mathrm{diag}(E_k, E_k)^*$ and $\mathrm{diag}(E_k, E_k)$, respectively, and use (11) to get that the matrix inequality

$$\left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\right]_k^* \begin{bmatrix} E_k^* & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} Z^*\bar{X}Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & E_k \end{bmatrix} \times \left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\right]_k - \begin{bmatrix} E_k^* & 0 \\ 0 & E_k^* \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & E_k \end{bmatrix} < -\beta I$$

holds, for every $k \geq 0$. Finally, use the definition of X to see that this last inequality is exactly

$$\left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\right]_k^* \left[\!\left[\begin{matrix} Z^*XZ & 0 \\ 0 & I \end{matrix}\right]\!\right]_k \left[\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\right]_k - \left[\!\left[\begin{matrix} X & 0 \\ 0 & I \end{matrix}\right]\!\right]_k < -\beta I \tag{12}$$

for each $k \geq 0$. This immediately implies that (10) is satisfied. □

The following corollary relates the infinite-dimensional linear matrix inequality to the pointwise properties of the system matrices.

Corollary 12: The following conditions are equivalent.

1) $\|C(I - ZA)^{-1}ZB + D\| < 1$ and $1 \notin \mathrm{spec}(ZA)$.
2) There exists a sequence of matrices $X_k > 0$, bounded above and below, such that the inequality

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}^* \begin{bmatrix} X_{k+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} - \begin{bmatrix} X_k & 0 \\ 0 & I \end{bmatrix} < 0$$

holds uniformly.

Proof: The result follows immediately from (12) in the proof of Theorem 11 using the fact that $(Z^*XZ)_k = X_{k+1}$. □

In the remainder of this section we will connect the result of Theorem 11 to the robust control object the structured singular value. In particular, we make use of the fact that the system function

$$\hat{G}(\Lambda) = C(I - \Lambda ZA)^{-1}\Lambda ZB + D$$

can be written as a linear fractional transformation on Λ. Using the results of the previous section, we regard $\Lambda := \mathrm{diag}(\lambda_0 I, \lambda_1 I, \lambda_2 I, \cdots)$ as a multidimensional frequency variable. The setup is illustrated in the Fig. 1. In particular, the induced norm of the system is less than one if and only if a performance result holds for the loop description of Fig. 1. Applying Theorem 7, we see that if $\|\hat{G}(\Lambda)\| < 1$ for all Λ corresponding to sequences of complex numbers on the complex unit disk, then the system G is contractive. We now state this in terms of the structured singular value with the standard perturbation class.

Define the set

$$\mathbf{\Delta}_0 = \{\Lambda \in \mathcal{L}(\ell_2)\colon \Lambda = \mathrm{diag}(\lambda_0 I, \lambda_1 I, \cdots), \lambda_k \in \mathbb{C}\}$$

and the set

$$\mathbf{\Delta} = \{\mathrm{diag}(\Lambda, \Phi) : \Lambda \in \Delta_0, \Phi \in \mathcal{L}(\ell_2)\}.$$

Hence given any element of $\mathbf{\Delta}$, the product

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Phi \end{bmatrix}$$

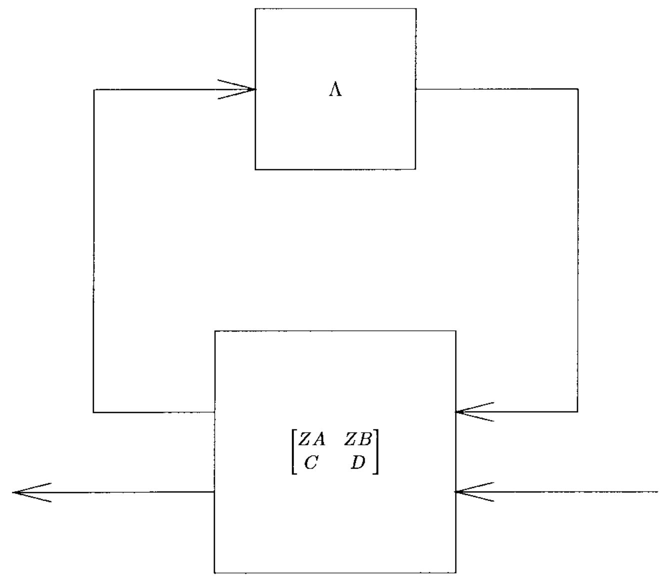

Fig. 1. The system function as an LFT.

is a map on $\ell_2 \oplus \ell_2$. For convenience we define the operator

$$M := \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}.$$

Now the structured singular value of M with respect to the set $\mathbf{\Delta}$ is

$$\mu(M) := \sup_{\Delta \in \mathcal{B}\mathbf{\Delta}} \mathrm{rad}(M\Delta)$$

where $\mathcal{B}\mathbf{\Delta}$ is the unit ball of the set $\mathbf{\Delta}$.

Now recall the definition of the set $\mathcal{X}$ in (8). Each X in $\mathcal{X}$ has the property

$$\Lambda X = X\Lambda$$

for all $\Lambda \in \mathbf{\Delta}_0$. Namely $\mathcal{X}$ is in the commutant of set $\mathbf{\Delta}_0$ and therefore forms a so-called D-scaling set. Having made these connections the next result follows from Theorem 11 by applying the standard techniques (see for instance [15]) of structured singular value theory.

Corollary 13: There exists $X \in \mathcal{X}$ satisfying (9) if and only if $\mu(M) < 1$.

This result says that the combined structure of the operator M and the set $\mathbf{\Delta}$ is μ-simple; namely the structured singular value in this case is equal to its standard upper bound. This is a nontrivial consequence of the structure of this particular setup.

Note that Lemma 10 is equivalent to the simpler result that the structured singular value is equal to its upper bound for the case where $\mathbf{\Delta}_0$ consists of operators of the form λI. Thus this corresponds to the well-known result that μ is equal to its upper bound for the case when the perturbation class $\mathbf{\Delta}$ consists of one full block and one scalar block. In the time-varying case, it is the special structure of M which allows us to achieve the much stronger result of Corollary 13.

In this section we have developed an analysis condition for evaluating the induced norm of an LTV system. In this framework the condition looks formally equivalent to LTI results and we will see in the next section that it leads directly to a simple synthesis result.

VI. Minimizing the ℓ_2-Induced Norm

Having developed the operator framework of the previous two sections to deal with LTV systems we now turn to the synthesis problem. That is, given a discrete LTV system, we would like to find a controller such that the closed-loop is contractive. In the results of the previous section we saw that, using the framework developed, it was possible to perform the analysis for the time-varying case by following directly the methods for the time-invariant case.

In this section, we solve the synthesis problem in the same way. Our methods are in the spirit of those employed in Packard [16] and Gahinet and Apkarian [9], and we shall see that once we have identified the analogous objects in our current framework, the conditions we obtain follow immediately from the LTI case. The development here most closely follows [9].

Let the system G be defined by the following state-space equations:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{1k} w_k + B_{2k} u_k \quad x_0 = 0 \\ z_k &= C_{1k} x_k + D_{11k} w_k + D_{12k} u_k \\ y_k &= C_{2k} x_k + D_{21k} w_k \end{aligned} \tag{13}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^{n_w}$, $u_k \in \mathbb{R}^{n_u}$, $z_k \in \mathbb{R}^{n_z}$, and $y_k \in \mathbb{R}^{n_y}$. We make the physical and technical assumption that the matrices A, B, C, and D are uniformly bounded functions of time. The only restrictions on this system are that the direct feedthrough term $D_{22} = 0$. This is a simple condition which is easy to ensure during implementation of such a system.

We suppose this system is being controlled by a controller K characterized by K

$$\begin{aligned} x_{k+1}^K &= A_k^K x_k^K + B_k^K y_k \\ u_k &= C_k^K x_k^K + D_k^K y_k. \end{aligned} \tag{14}$$

where $x_k^K \in \mathbb{R}^m$. The connection of G and K is shown in Fig. 2. Since $D_{22} = 0$, this interconnection is always well-posed.

We write the realization of the closed-loop system as

$$\begin{aligned} x_{k+1}^L &= A_k^L x_k + B_k^L w_k \\ z_k &= C_k^L x_k + D_k^L w_k \end{aligned} \tag{15}$$

where x_k contains the combined states of G and K, and A_k^L, B_k^L, C_k^L, and D_k^L are appropriately defined. Here $A^L \in \mathbb{R}^{(n+m)\times(n+m)}$, where n is the number of states of G and m is the number of states of K.

We are only interested in controllers K that both stabilize G and provide acceptable performance as measured by the induced norm of the map $w \mapsto z$. The following definition expresses our synthesis goal.

Definition 14:

A controller K is an *admissible synthesis* for G in Fig. 2, if $1 \notin \mathrm{spec}(ZA^L)$ and the closed-loop performance inequality $\|w \mapsto z\|_{\ell_2 \to \ell_2} < 1$ is achieved.

Hence, recalling Proposition 6 we are requiring the closed-loop system defined by (15) be exponentially stable, in addition to being strictly contractive.

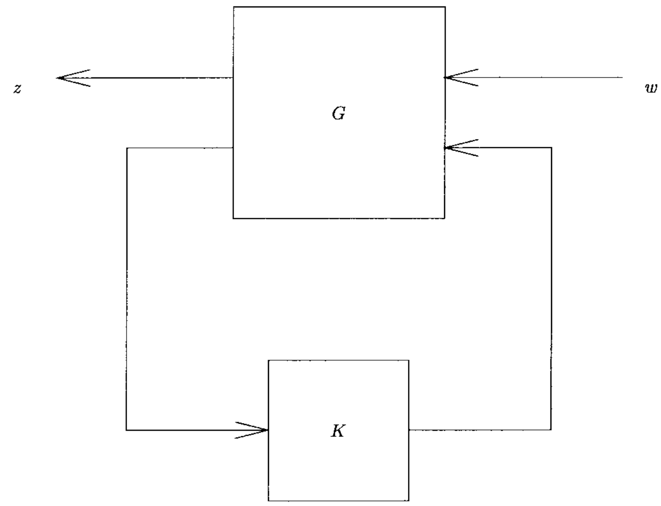

Fig. 2. Closed-loop system.

We can parameterize the closed-loop relation in terms of the controller realization as follows. First we make the following definitions:

$$\bar{A} = \left[\!\left[\begin{matrix} A & 0 \\ 0 & 0_{\ell_2}^{m\times m} \end{matrix}\right]\!\right] \qquad \bar{B} = \left[\!\left[\begin{matrix} B_1 \\ 0_{\ell_2}^{m\times n_w} \end{matrix}\right]\!\right]$$

$$\bar{C} = \left[\!\left[\begin{matrix} C_1 & 0_{\ell_2}^{n_z\times m} \end{matrix}\right]\!\right] \qquad \underline{C} = \left[\!\left[\begin{matrix} 0 & I_{\ell_2}^{m\times m} \\ C_2 & 0 \end{matrix}\right]\!\right]$$

$$\underline{B} = \left[\!\left[\begin{matrix} 0 & B_2 \\ I_{\ell_2}^{m\times m} & 0 \end{matrix}\right]\!\right] \qquad \underline{D}_{12} = \left[\!\left[\begin{matrix} 0_{\ell_2}^{n_z\times m} & D_{12} \end{matrix}\right]\!\right]$$

$$\underline{D}_{21} = \left[\!\left[\begin{matrix} 0_{\ell_2}^{m\times n_w} \\ D_{21} \end{matrix}\right]\!\right].$$

Observe that these operators depend only on the realization of the system G and are entirely independent of K. Now group the controller realization together into the block-diagonal operator J defined by

$$J := \left[\!\left[\begin{matrix} A^K & B^K \\ C^K & D^K \end{matrix}\right]\!\right]. \tag{16}$$

From these definitions we now see that the closed-loop parameterization can be written as

$$\begin{aligned} A^L &= \bar{A} + \underline{B} J \underline{C} \quad & B^L &= \bar{B} + \underline{B} J \underline{D}_{21} \\ C^L &= \bar{C} + \underline{D}_{12} J \underline{C} \quad & D^L &= D_{11} + \underline{D}_{12} J \underline{D}_{21} \end{aligned} \tag{17}$$

where each operator is block-diagonal. The crucial property of this parameterization is that each operator depends *affinely* on the controller realization J.

The following result makes use of the affine expressions for the closed loop to give a test for whether a given controller is admissible. In order to state this result, define the following operators:

$$\begin{aligned} P &:= \left[\underline{B}^* \quad 0_{\ell_2}^{(m+n_u)\times(n+m)} \quad 0_{\ell_2}^{(m+n_u)\times n_w} \quad \underline{D}_{12}^*\right] \\ Q &:= \left[0_{\ell_2}^{(m+n_u)\times(n+m)} \quad \underline{C} \quad \underline{D}_{21} \quad 0_{\ell_2}^{(m+n_u)\times n_z}\right]. \end{aligned}$$

Further let $\mathcal{X}^L$ be the set of strictly positive block-diagonal operators, defined as in (8), with block structure corresponding to that of A^L. Then, for $X \in \mathcal{X}^L$, define

$$H_X := \begin{bmatrix} -Z^* X^{-1} Z & \bar{A} & \bar{B} & 0 \\ \bar{A}^* & -X & 0 & \bar{C}^* \\ \bar{B}^* & 0 & -I & D_{11}^* \\ 0 & \bar{C} & \bar{D}_{11} & -I \end{bmatrix}. \tag{18}$$

The following result gives the desired test for admissibility.

Lemma 15: The controller K described by the block-diagonal operator J is admissible if and only if there exists $X \in \mathcal{X}^L$ such that

$$H_X + Q^* J^* P + P^* J Q < 0. \tag{19}$$

Proof: We first apply the Schur complement to Theorem 11, to see that, with the controller K in place, the closed-loop performance criterion is satisfied if and only if there exists $X \in \mathcal{X}^L$ such that

$$\begin{bmatrix} -Z^* X^{-1} Z & A^L & B^L & 0 \\ A^{L*} & -X & 0 & C^{L*} \\ B^{L*} & 0 & -I & D^{L*} \\ 0 & C^L & D^L & -I \end{bmatrix} < 0. \tag{20}$$

We can now substitute into this equation the expressions in (17) for the closed-loop realization in terms of J, the controller realization. This immediately gives the desired result. □

Note that (20) can be expressed equivalently as

$$\begin{bmatrix} -X^{-1} & ZA^L & ZB^L & 0 \\ A^{L*} Z^* & -X & 0 & C^{L*} \\ B^{L*} Z^* & 0 & -I & D^{L*} \\ 0 & C^L & D^L & -I \end{bmatrix} < 0.$$

This expression clearly shows the parallel between this result and the corresponding result in the time-invariant case, the former being derived from the latter by formally replacing the A-matrix and B-matrix by ZA and ZB. However, we will work with (20), since it consists solely of block-diagonal operators.

Note that H_X, P, and Q depend solely on the system G and are independent of J. In order to make use of the above lemma, we use the following important technical lemma, from [16] and [9].

Lemma 16: Given a symmetric matrix T, matrices E and F, and a number $\beta > 0$, there exists a matrix Θ that satisfies

$$T + E^* \Theta^* F + F^* \Theta E < -\beta I$$

if and only if

$$W_E^* T W_E < -\beta I$$
$$W_F^* T W_F < -\beta I$$

where $\operatorname{Im} W_F = \operatorname{Ker} F$, $\operatorname{Im} W_E = \operatorname{Ker} E$, $W_F^* W_F = I$ and $W_E^* W_E = I$.

In order to state the next lemma, and take advantage of this parameterization, we define the sequences of matrices U_{1k}, U_{2k}, V_{1k}, and V_{2k} such that

$$\operatorname{Im} \begin{bmatrix} V_{1k} \\ V_{2k} \end{bmatrix} = \operatorname{Ker} [B_{2k}^* \quad D_{12k}^*]$$

$$\operatorname{Im} \begin{bmatrix} U_{1k} \\ U_{2k} \end{bmatrix} = \operatorname{Ker} [C_{2k} \quad D_{21k}]$$

and $V_k^* V_k = I$ and $U_k^* U_k = I$, for each k. Clearly, from these matrix sequences we can construct block-diagonal operators U_1, U_2, V_1, and V_2. In general the blocks here may not all have the same number of columns but will all have the same number of rows. However, in the general case it is straightforward to show that this operator is well-defined, and the matrix product RU_1 is block-diagonal for any R with compatible block-diagonal structure. However, for notational simplicity in the following we consider only the case when all blocks are the same size, although in fact the formulas we derive are valid in the general setting.

Define

$$W_P = \begin{bmatrix} V_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{\ell_2}^n & 0 & 0 \\ 0 & 0 & I_{\ell_2}^m & 0 \\ 0 & 0 & 0 & I_{\ell_2}^{n_z} \\ V_2 & 0 & 0 & 0 \end{bmatrix}$$
$$W_Q = \begin{bmatrix} 0 & I_{\ell_2}^n & 0 & 0 \\ 0 & 0 & I_{\ell_2}^m & 0 \\ U_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ U_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\ell_2}^{n_z} \end{bmatrix}. \tag{21}$$

It is apparent that $\operatorname{Im} W_P = \operatorname{Ker} P$ and $\operatorname{Im} W_Q = \operatorname{Ker} Q$, and furthermore $W_Q^* W_Q = I$ and $W_P^* W_P = I$. We are now in a position to prove the following major lemma.

Lemma 17: There exists a synthesis for G if and only if there exists a block-diagonal operator $X \in \mathcal{X}^L$ such that

$$W_P^* H_X W_P < 0 \quad \text{and} \quad W_Q^* H_X W_Q < 0 \tag{22}$$

where W_Q and W_P are defined in (21) and H_X is defined in (18).

Proof: We start by invoking Lemma [15], which states that a controller K is admissible if and only if there exists a block-diagonal operator $X \in \mathcal{X}^L$ such that

$$H_X + Q^* J^* P + P^* J Q < 0. \tag{23}$$

Applying Propositions 3 and 4, this is equivalent to

$$[\![H_X]\!] + [\![Q]\!]^* J^* [\![P]\!] + [\![P]\!]^* J [\![Q]\!] < 0$$

since the block structures are compatible and $[\![J]\!] = J$. Hence this operator inequality holds if and only if there exists $\beta > 0$ such that

$$[\![H_X]\!]_k + [\![Q]\!]_k^* J_k^* [\![P]\!]_k + [\![P]\!]_k^* J_k [\![Q]\!]_k < -\beta I$$

for all $k \geq 0$. For each k, this is simply a matrix equation, and we can apply Lemma 16. Further, by construction

$$\operatorname{Ker} [\![P]\!]_k = \operatorname{Im} [\![W_P]\!]_k \quad \text{and} \quad \operatorname{Ker} [\![Q]\!]_k = \operatorname{Im} [\![W_Q]\!]_k$$

and hence the above operator inequality holds if and only if there exists $\beta > 0$ such that

$$[\![W_P]\!]_k [\![H_X]\!]_k [\![W_P]\!]_k < -\beta I$$
$$[\![W_Q]\!]_k [\![H_X]\!]_k [\![W_Q]\!]_k < -\beta I$$

for all $k \geq 0$. Now applying Proposition 4 again, this is equivalent to the desired result. □

Having proved the last lemma we have our system in exactly the same form as the LTI paper of [9], but now we have block-diagonal operators in place of matrices. We can therefore use manipulations that are formally equivalent.

One problem with the result of Lemma 17 is that the operator inequalities derived are not affine in X, since both X and X^{-1} appear in the operator H_X. We would therefore like to express them in an affine form.

To progress with this task, we must examine closely the form of these inequalities and the block-diagonal operator X which appears in them. Clearly, each block X_k has dimension $(n+m)\times(n+m)$. Given such a block-diagonal X, define the block-diagonal operators R and S via

$$X = \left[\!\!\left[\begin{matrix} S & N \\ N^* & ? \end{matrix}\right]\!\!\right] \qquad X^{-1} = \left[\!\!\left[\begin{matrix} R & L \\ L^* & ? \end{matrix}\right]\!\!\right] \tag{24}$$

where $R_k, S_k \in \mathbb{R}^{n\times n}$ and $L_k, N_k \in \mathbb{R}^{n\times m}$.

We will show that X satisfies (22) if R and S satisfy particular linear matrix inequalities. We will also see that if there exist R and S satisfying these matrix inequalities, then X can be constructed from them such that the inequalities in (22) hold. This will therefore give us convex necessary and sufficient conditions for the existence of an admissible synthesis for G.

In order to accomplish this we have the following lemma, based on [16], which states when it is possible to construct a strictly positive operator X, satisfying (24), from two operators $R > 0$ and $S > 0$.

Lemma 18: Suppose $R > 0$ and $S > 0$ are block-diagonal operators with entries $R_k, S_k \in \mathbb{R}^{n\times n}$ and that the integer $m \geq n$. Then there exists an operator $X > 0$, satisfying (24), with entries $X_k \in \mathbb{R}^{(n+m)\times(n+m)}$, if and only if

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0.$$

The proof of this is nearly identical to its matrix version found in [16] and so we do not include it here.

The following theorem transforms the inequalities in (22) to a condition that only depends on the plant data and is independent of m, the controller state dimension; more importantly these conditions are convex.

Theorem 19: There exists an admissible synthesis K for G, with state dimension $m \geq n$, if and only if there exist block-diagonal operators $R > 0$ and $S > 0$ satisfying:

1)
$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} ARA^* - Z^*RZ & ARC_1^* & B_1 \\ C_1RA^* & C_1RC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \times \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0$$

2)
$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*Z^*SZA - S & A^*Z^*SZB_1 & C_1 \\ B_1^*Z^*SZA & B_1^*Z^*SZB_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0$$

3)
$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0$$

where the operators N_R, N_S satisfy

$$\operatorname{Im} N_R = \operatorname{Ker}[B_2^* \quad D_{12}]^* \quad N_R^*N_R = I$$
$$\operatorname{Im} N_S = \operatorname{Ker}[C_2 \quad D_{21}] \quad N_S^*N_S = I.$$

Proof: By Lemma 17 there exists an admissible synthesis if and only if there exists an appropriately dimensioned block-diagonal operator $X > 0$ such that inequalities

$$W_P^*H_XW_P < 0 \quad \text{and} \quad W_Q^*H_XW_Q < 0 \tag{25}$$

hold. It is therefore sufficient to show that the existence of such an X, with the state dimension $m \geq n$, is equivalent to conditions 1)–3) in the theorem statement.

(Only If): First assume that $X \in \mathcal{X}^L$ satisfies the conditions in (25), and define R and S as in (24). Now examining the partition of H_X and W_P it is straightforward to demonstrate that $W_P^*H_XW_P < 0$ is satisfied if and only if

$$\begin{bmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ V_2 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} -Z^*RZ & AE & B_1 & 0 \\ E^*A^* & -X & 0 & E^*C_1^* \\ B_1^* & 0 & -I & D_{11}^* \\ 0 & C_1E & D_{11} & -I \end{bmatrix} \times \begin{bmatrix} V_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ V_2 & 0 & 0 \end{bmatrix} < 0$$

where $E = [\![I \ 0]\!]$. Applying the Schur complement formula so as to invert X, and permuting the blocks the above condition is equivalent to

$$\begin{bmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} ARA^* - Z^*RZ & ARC_1^* & B_1 \\ C_1RA^* & C_1RC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \times \begin{bmatrix} V_1 & 0 \\ V_2 & 0 \\ 0 & I \end{bmatrix} < 0 \tag{26}$$

where using the structure of $\bar{A}$ is crucial. From the definition of N_R, this implies that 1) holds. A similar argument starting with $W_Q^*H_XW_Q < 0$ shows that 2) holds with S defined from X as in (24). Finally 3) must hold by Lemma 18 and the definition of R and S from X.

(If): Suppose there exist block-diagonal operators R and S satisfying 1)–3). Then by invoking Lemma 18, with m set to be n, there exists a block-diagonal operator $X \in \mathcal{X}^L$ that satisfies (24). Now routine manipulations, reversing the "only if" argument, show that this X must satisfy the inequalities in (25). □

It is interesting to note that condition 1) in the above theorem can also be written as

$$\begin{bmatrix} \bar{N}_R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} ZARA^*Z^* - R & ZARC_1^* & ZB_1 \\ C_1RA^*Z^* & C_1RC_1^* - I & D_{11} \\ B_1^*Z^* & D_{11}^* & -I \end{bmatrix} \times \begin{bmatrix} \bar{N}_R & 0 \\ 0 & I \end{bmatrix} < 0$$

where $\operatorname{Im} \bar{N}_R = \operatorname{Ker}[B_2^*Z^* \quad D_{12}^*]$ and $\bar{N}_R^*\bar{N}_R = I$. This makes the correspondence with the time-invariant case and the formulas of [9] clear; formally one can simply replace the A-matrix by ZA and the B-matrix by ZB in the latter to arrive at the former.

Note also that if we define the sequences

$$\operatorname{Im} NR_k = \operatorname{Ker}[B_{2k}^* \quad D_{12k}^*] \quad \operatorname{Im} N_{Sk} = \operatorname{Ker}[C_{2k} \quad D_{21k}]$$

which are directly related to N_R and N_S, then the conditions of Theorem 19 are easily seen to be equivalent to the existence of $\beta > 0$ such that

$$\begin{bmatrix} N_{R_k} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_k R_k A_k' - R_{k+1} & A_k R_k C_{1k}' & B_{1k} \\ C_{1k} R_k A_k' & C_{1k} R_k C_{1k}' - I & D_{11k} \\ B_{1k}' & D_{11k}^* & -I \end{bmatrix} \times \begin{bmatrix} N_{R_k} & 0 \\ 0 & I \end{bmatrix} < -\beta I$$

$$\begin{bmatrix} N_{S_k} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A_k' S_{k+1} A_k - S_k & A_k' S_{k+1} B_{1k} & C_{1k} \\ B_{1k}' S_{k+1} A_k & B_{1k}^* S_{k+1} B_{1k} - I & D_{11k}^* \\ C_{1k} & D_{11k} & -I \end{bmatrix} \times \begin{bmatrix} N_{S_k} & 0 \\ 0 & I \end{bmatrix} < -\beta I$$

$$\begin{bmatrix} R_k & I \\ I & S_k \end{bmatrix} \geq 0$$

for all $k \geq 0$. This gives a recursive matrix form of the solution.

We now briefly outline a procedure for finding a synthesis given operators R and S satisfying Theorem 19. Start by constructing a block-diagonal operator X, which must exist by Lemma 18, such that the equations in (24) hold. Then by Lemma 17 the operator X must satisfy (22). Therefore, there exists a block-diagonal operator J satisfying (23). The controller specified by

$$J_k = \begin{bmatrix} A_k^K & B_k^K \\ C_k^K & D_k^K \end{bmatrix}$$

will now be an admissible synthesis for G. All of the above steps are convex but infinite-dimensional computations, and thus in general may be hard to carry out. However, in the next section we develop finite-dimensional conditions for which this procedure is in general feasible; see [9] or [16] for more details on carrying out such a controller construction from R and S.

We have thus derived a complete solution to the induced ℓ_2-norm synthesis problem for discrete LTV systems, simply by following the methodology used in the time-invariant case in [9], [16] and making use of the mathematical tools developed in this paper. The solution derived holds for general systems in the same way as the LTI solution; there are no requirements that D_{12} or D_{21} be full rank or that D_{11} be zero. Further, this solution has the important property of being convex. This offers not only powerful computational properties, but also gives insight into the structure of the solution.

The next section gives a particularly simple derivation of the solution for periodic systems by making use of convexity.

VII. Periodic Systems and Finite-Dimensional Conditions

The analysis and synthesis conditions stated in Theorems 11 and 19 are in general infinite-dimensional. However, there are two important cases in which they reduce to finite-dimensional convex problems. The first is when one is only interested in behavior on the finite horizon. In this case the matrix sequences A_k, B_k, C_k, and D_k would be chosen to be zero for $k \geq N$ the length of the horizon. Thus the associated synthesis and analysis inequalities immediately reduce to finite-dimensional conditions. The second major case that reduces occurs when the system G is periodic and a periodic controller K is sought. Developing these conditions is the purpose of this section.

An operator P on ℓ_2 is said to be q-periodic if

$$Z^q P = P Z^q$$

namely it commutes with q shifts. Throughout the sequel we fix $q \geq 1$ to be some integer. With this definition we can now prove the main technical result of this section.

Theorem 20: Suppose A, B, C, and D are q-periodic operators and that $X \in \mathcal{X}$ and satisfies (9). Then there exists a q-periodic operator $X_{\text{per}} \in \mathcal{X}$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X_{\text{per}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X_{\text{per}} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{27}$$

The theorem says that a solution exists to the performance inequality if and only if a periodic solution exists. Note that the proof below amounts to taking an average of a sequence of solutions to (9) where each is constructed from X by shifting. A similar averaging technique is used in [7], in the context of time-varying control analysis.

Proof: By assumption $X \in \mathcal{X}$ satisfies (9). Therefore, there exist numbers $\alpha > 0$ and $\beta > 0$, such that $X > \alpha I$ and

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < -\beta I. \tag{28}$$

For convenience let $L = \left[\begin{smallmatrix} Z^q & 0 \\ 0 & Z^q \end{smallmatrix}\right]$, and observe that $L^* L = I$ since $Z^* Z = I$. For an integer $k \geq 0$, pre- and postmultiply the above inequality by $(L^*)^k$ and L^k, respectively, to get

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} (Z^*)^{qk} X Z^{qk} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} (Z^*)^{qk} X Z^{qk} & 0 \\ 0 & I \end{bmatrix} < -\beta I \tag{29}$$

which easily follows from the facts that Z^q commutes with A, B, C, and D.

From (29) define M and Q_k in the obvious way to write the inequality more compactly as

$$M^* Q_k M - Q_k < -\beta I. \tag{30}$$

Notice that $Q_k > \alpha I$ and $\|Q_k\| \leq \max\{\|X\|, 1\}$ since $Z^* Z = I$. Now define Y_N to be the finite average

$$Y_N = \frac{1}{N} \sum_{k=0}^{N-1} Q_k, \qquad \text{for } N \geq 1. \tag{31}$$

Since the sequence Q_k is bounded, so is the sequence Y_N. Thus there exist a subsequence Y_{N_k} and an operator Y to which the subsequence converges in the weak operator topology (see for instance [10] for this property). Without loss of generality we assume

$$\lim_{N\to\infty} Y_N \overset{\text{weak}}{=} Y.$$

Clearly Y must be self-adjoint and satisfy $Y \geq \alpha I$ since each Y_N has these two properties. Also Y has the form

$$Y = \begin{bmatrix} X_{\text{per}} & 0 \\ 0 & I \end{bmatrix}$$

where $X_{\text{per}} \in \mathcal{X}$ because, for each N, the operator Y_N has this form, and Y is the weak limit of this sequence. To complete the proof we must show that $Z^q X_{\text{per}} = X_{\text{per}} Z^q$ and X_{per} satisfies (27).

To show the former we demonstrate that $L^* Y L = Y$: from (31) and the definition of Q_k in (30) it is easy to verify that

$$L^* Y_N L - Y_N = \frac{1}{N}\{Q_N - Q_0\}.$$

Therefore $\lim_{N\to\infty}\{L^* Y_N L - Y_N\} = 0$, and by the properties of weak convergence it follows that $L^* Y L - Y = 0$.

Finally to show that X_{per} satisfies (27), we use linearity of (29) and the definition of Y_N to see that

$$M^* Y_N M - Y_N < -\beta I$$

holds for each $N \geq 1$. Again, it is routine to show using the definition of weak convergence that this necessarily means $M^* Y M - Y \leq -\beta I$, which immediately implies that (27) is satisfied. □

Before stating the next result we require some additional notation. Suppose Q is a q-periodic block-diagonal operator, then we define $\tilde{Q}$ to be the first period truncation of Q, namely

$$\tilde{Q} := \begin{bmatrix} Q_0 & & 0 \\ & \ddots & \\ 0 & & Q_{q-1} \end{bmatrix}$$

which is a matrix. Also define the cyclic shift matrix $\tilde{Z}$, for $q \geq 2$, by

$$\tilde{Z} = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & \ddots & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}$$

such that

$$\tilde{Z}^* \tilde{Q} \tilde{Z} = \begin{bmatrix} Q_1 & & & 0 \\ & \ddots & & \\ & & Q_{q-1} & \\ 0 & & & Q_0 \end{bmatrix}.$$

For $q = 1$ set $\tilde{Z} = I$. Also define the truncation of the set $\mathcal{X}$, defined in (8), by

$$\tilde{\mathcal{X}} := \{\tilde{X} : X \in \mathcal{X}\}.$$

Using these new definitions we have the following corollary of Theorem 20 and Theorem 11.

Corollary 21: Suppose A, B, C, and D are q-periodic operators. The following conditions are equivalent.

1) $\|C(I - ZA)^{-1}ZB + D\| < 1$ and $1 \notin \operatorname{spec}(ZA)$.
2) There exists a matrix $\tilde{X} \in \tilde{\mathcal{X}}$ such that

$$\begin{bmatrix} \tilde{Z}\tilde{A} & \tilde{Z}\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^* \begin{bmatrix} \tilde{X} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{Z}\tilde{A} & \tilde{Z}\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} - \begin{bmatrix} \tilde{X} & 0 \\ 0 & I \end{bmatrix} < 0. \tag{32}$$

Thus this corollary gives a finite-dimensional convex condition for determining the ℓ_2-induced norm of a periodic system of the form in (1). This condition can be checked using various convex programming techniques; see for example [5] for a synopsis of such methods.

In the case of a periodic system G the infinite-dimensional synthesis conditions of the last section can be extended to obtain finite-dimensional ones. This gives the following synthesis result.

Theorem 22: Suppose A, B, C, and D are q-periodic operators. There exists an admissible synthesis K for G, with state dimension $m \geq n$, if and only if there exist block-diagonal matrices $\tilde{R} > 0$ and $\tilde{S} > 0$ satisfying:

1)
$$\begin{bmatrix} \tilde{N}_R & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \tilde{A}\tilde{R}\tilde{A}^* - \tilde{Z}^*\tilde{R}\tilde{Z} & \tilde{A}\tilde{R}\tilde{C}_1^* & \tilde{B}_1 \\ \tilde{C}_1\tilde{R}\tilde{A}^* & \tilde{C}_1\tilde{R}\tilde{C}_1^* - I & \tilde{D}_{11} \\ \tilde{B}_1^* & \tilde{D}_{11}^* & -I \end{bmatrix} \times \begin{bmatrix} \tilde{N}_R & 0 \\ 0 & I \end{bmatrix} < 0$$

2)
$$\begin{bmatrix} \tilde{N}_S & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \tilde{A}^*\tilde{Z}^*\tilde{S}\tilde{Z}\tilde{A} - \tilde{S} & \tilde{A}^*\tilde{Z}^*\tilde{S}\tilde{Z}\tilde{B}_1 & \tilde{C}_1 \\ \tilde{B}_1^*\tilde{Z}^*\tilde{S}\tilde{Z}\tilde{A} & \tilde{B}_1^*\tilde{Z}^*\tilde{S}\tilde{Z}\tilde{B}_1 - I & \tilde{D}_{11}^* \\ \tilde{C}_1 & \tilde{D}_{11} & -I \end{bmatrix} \times \begin{bmatrix} \tilde{N}_S & 0 \\ 0 & I \end{bmatrix} < 0$$

3)
$$\begin{bmatrix} \tilde{R} & I \\ I & \tilde{S} \end{bmatrix} \geq 0$$

where the operators $\tilde{N}_R$, $\tilde{N}_S$ satisfy

$$\operatorname{Im}\tilde{N}_R = \operatorname{Ker}[\tilde{B}_2^* \quad \tilde{D}_{12}^*] \quad \tilde{N}_R^*\tilde{N}_R = I$$
$$\operatorname{Im}\tilde{N}_S = \operatorname{Ker}[\tilde{C}_2 \quad \tilde{D}_{21}] \quad N_S^* N_S = I.$$

This theorem reduces the existence of a synthesis for G to a matrix condition. To prove the theorem one first shows that there exist operators R and S satisfying Theorem 19 if and only if there exist q-periodic solutions R_{per} and S_{per}. This is done using the same averaging argument that was employed in the proof of Theorem 20. Theorem 22 follows immediately. Solutions $\tilde{R}$ and $\tilde{S}$ above can be used to construct a q-periodic controller K and therefore the theorem also gives the result that a synthesis exists for G if and only if a q-periodic synthesis exists.

VIII. Conclusion

In this paper we have developed a new operator theoretic framework for the treatment of time-varying systems. The key feature of this new setting is that LTV systems viewed in the framework look formally equivalent to LTI systems. Indeed state-space matrices are replaced by block-diagonal operators.

We have developed tools for effectively working in this environment and shown how to apply this machinery to solve general versions of the H_∞ analysis and synthesis problems for time-varying systems. The results appear similar to those for LTI systems, except that in the general case they are infinite-dimensional convex problems. In the case of periodic systems it was seen that these conditions reduced to being finite-dimensional.

Since the approach developed in this paper establishes a strong connection with LTI analysis techniques, we believe that it may find wider application in time-varying systems analysis in the context of robust control.

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