Error-Bounds for Balanced Model-Reduction of Linear Time-Varying Systems

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***Abstract*—Error-bounds are developed for balanced truncation of linear time-varying systems, leading to an extension of the "twice the sum of the tail" formulas, well known in the time-invariant case. The approach relies on an operator-theoretic framework for analysis of linear time-varying systems. This provides a multivariable notion of frequency for such systems, which are thus characterized by rational functions of many complex variables, allowing the problem to be formulated in the linear-fractional framework. Using a time-varying version of standard necessary conditions for reduced-order modeling, based on convex operator inequalities, we show that these error-bounds for balanced truncation are related to the closest possible reduced-order modeling error in a sense which parallels the time-invariant case.**

***Index Terms*—Balanced truncation, model reduction, time-varying systems.**

I. Introduction

An important problem in control is that of reduced-order modeling of dynamical systems. Starting with a high-dimensional or partial-differential model of a complex physical system, one would like to construct a simplified model, with the goal of increased computational and mathematical tractability. Applications include simulation, design optimization and stability analysis as well as control synthesis. In the latter case, an example includes induced-norm synthesis, where computation time typically grows faster than $O(n^3)$, where n is the state-dimension of the model.

In this paper, we consider model reduction of systems described by stable discrete-time linear processes of the form

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k, \qquad x_0 = 0 \\ z_k &= C_k x_k + D_k w_k. \end{aligned} \tag{1}$$

Here, the state is $x_k \in \mathbb{R}^n$, the input $w_k \in \mathbb{R}^{n_w}$, and the output $z_k \in \mathbb{R}^{n_z}$. We further develop results that hold when the dimension of the state x_k varies with k. As is standard, if this system is stable, then corresponding to it there is a bounded linear operator G on the space of square-summable sequences ℓ_2, where $G: w \mapsto z$.

Central to the area of model reduction is the question of error analysis. Given a system model G with state dimension n, we would like to construct a *reduced-order* system G_r of dimension $r \ll n$ which minimizes an appropriate error measure between G and G_r. The error measure we consider in this paper is the induced norm, defined for an operator H by

$$\|H\| = \sup_{w \in \ell_2, w \neq 0} \frac{\|Hw\|}{\|w\|}.$$

In this case, we would like to minimize $\|G - G_r\|$ for a fixed r.

The operator G has a triangular structure, and in the case when the matrices A, B, C, D are independent of time k, the operator G is Toeplitz; that is, it has the structure

$$G = \begin{bmatrix} g_0 & & & & \\ g_1 & g_0 & & & \\ g_2 & g_1 & g_0 & & \\ g_3 & g_2 & g_1 & g_0 & \\ \vdots & & & & \ddots \end{bmatrix}$$

where

$$g_i = \begin{cases} D, & \text{if } i = 0 \\ CA^{i-1}B, & \text{otherwise.} \end{cases}$$

Corresponding to G there is an associated operator Γ^G with a Hankel structure of the form

$$\Gamma^G = \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & \cdots \\ g_1 & g_2 & g_3 & \cdots & \\ g_2 & g_3 & & & \\ g_3 & & & & \\ \vdots & & & & \end{bmatrix}.$$

The operator Γ^G is called the Hankel operator corresponding to G. Given a linear time-invariant operator G, the number of states of the corresponding dynamical system (1), and hence the dimension of the A matrix, is equal to the rank of Γ^G.

For model reduction, we would like to construct a low-rank Hankel operator corresponding to G by approximating Γ^G in some appropriate sense. Unfortunately one cannot simply use a singular-value decomposition of Γ^G since this will not preserve the Hankel structure. In [1], it was shown that good approximants could be constructed with an upper bound on the error in the Hankel norm, and Glover [2] gave a characterization of all optimal Hankel-norm approximants of a given G, together with an upper bound on the induced-norm error.

Balanced truncation is a method for constructing a reduced-order approximant to G based on a particular factorization of Γ^G. Let the singular values of the operator Γ^G be $\sigma_1 > \sigma_2 > \cdots > \sigma_k > 0$, with σ_i having multiplicity d_i, so that $\sum d_i = \operatorname{rank}(\Gamma^G)$. The σ_i are known as the *Hankel singular values* of

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G. Roughly speaking, for linear time-invariant systems the balanced truncation method projects the dynamics onto the subspace of the state-space corresponding to the first $n-r$ Hankel singular values. The induced-norm error is then less than twice the sum of the Hankel singular values which have been truncated, excluding multiplicities.

Time-varying systems: Our focus in this paper is on time-varying systems, that is those in which the matrices $A,\ B,\ C,\ D$ depend on k. In this case a very similar theory holds, and there is a *sequence* of Hankel operators Γ_k^G corresponding to the operator G. We shall show that the error made in balanced truncation is bounded by twice the sum of the truncated Hankel singular values in an appropriate generalized sense.

A. Previous Work

The balanced realization was introduced by Mullis and Roberts [3] for linear time-invariant (LTI) systems, and was first proposed for model reduction by Moore [4], with further developments in [5], [6]. For LTI continuous-time systems, an *a priori* error bound in the H_∞ norm was found by Glover [2] and Enns [7], and the corresponding bounds for LTI discrete-time systems were presented by Al-Saggaf and Franklin [8] and Hinrichsen and Pritchard [9]. These results were further generalized to uncertain systems in the linear-fractional framework by Beck *et al.* [10]–[12].

For linear time-varying (LTV) systems, the use of balanced truncation for model reduction of continuous time systems has been developed in [13]–[18]. Analytic systems were studied in [17] and [18], uniform controllability and observability were used in [14]–[16], and an approach called μ balancing making use of Riccati differential equations for computation was presented in [13].

Discrete-time LTV systems have been treated in [19]–[21], with the periodic case studied in [21], and the discrete two-dimensional spatially-varying case treated in [22]. Balancing methods have also been proposed for model reduction of jump-linear systems in [23]. The work in these papers generalizes the standard time-invariant results by utilizing time-varying versions of the standard controllability and observability Gramians.

A closely related problem, that of optimal approximation in the Hankel-norm, has also been considered [24]; the relationship between Hankel operators for time-varying systems and balanced truncation is discussed in [19]. Balanced truncation of a related class of systems called linear parameter-varying (LPV) systems has been considered in [25] and [26] for the continuous-time case.

An important consequence of the work on balanced truncation of linear time-invariant systems was the development of the "twice the sum of the tail" formulas for the error bound. Previous work has not addressed the question of whether similar formulae exist in the general time-varying case, and it is this problem which is the focus of this paper.

This paper is a longer version of [27], where we proposed a new approach to this problem for periodically time-varying and aperiodic LTV systems. In the current paper we focus on aperiodic systems, and the main contribution is the development of upper-bounds on the error, in the induced two-norm, between the truncated and the original system. Since [27], related error-bound results have also been developed for periodically time-varying systems in [28], [29].

We make use of the generalized notion of balancing which was presented in [12], [9], where linear matrix inequalities (LMIs) are used to define generalized Gramians with similar properties to the usual controllability and observability Gramians. This generalized notion of Gramians allowed for model reduction of uncertain systems described by linear fractional transformations (LFTs), and although in this paper we shall focus on systems without uncertainty a very similar philosophy will be adopted.

The main mathematical methodology will be via an operator theoretic framework for analysis of time-varying systems [30], which provides a multivariable notion of frequency for such systems. By formulating LTV systems as a linear-fractional transformation over such frequency variables, we can bring to bear very similar machinery to that used for the uncertain case to the LTV case. There are many interesting links and connections between realization, model reduction, and Hankel operators for time-varying systems, and these ideas have been developed in several different contexts; see, for example, [31]–[36].

B. Mathematical Preliminaries

The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. Given two Hilbert spaces E and F we denote the space of bounded linear operators mapping E to F by $\mathcal{L}(E,\ F)$, and shorten this to $\mathcal{L}(E)$ when E equals F. If $X \in \mathcal{L}(E,\ F)$ we denote the induced norm of X by $\|X\|$ and the adjoint of X by X^*. For $X \in \mathcal{L}(E)$, we denote its spectrum by $\operatorname{spec}(X)$, defined by

$$\operatorname{spec}(X) = \{\lambda \in \mathbb{C} | \lambda I - X \text{ is singular}\}.$$

When $X \in \mathcal{L}(E)$ is self adjoint we use $X < 0$ to mean that there exists a number $\alpha > 0$ such that for all nonzero $x \in E$ the inequality

$$\langle x,\ Xx\rangle < -\alpha \|x\|^2$$

holds, where $\langle \cdot,\ \cdot \rangle$ denotes the inner product and $\|\cdot\|$ denotes the corresponding norm on E.

The main Hilbert space of interest in this paper is denoted by $\ell_2(J)$ where J is a sequence of Euclidean spaces

$$J = (\mathbb{R}^{n_0},\ \mathbb{R}^{n_1},\ \mathbb{R}^{n_2},\ \ldots).$$

It consists of elements $x = (x_0,\ x_1,\ x_2,\ \ldots)$, with each $x_k \in J_k$, which have a finite two-norm $\|x\|$ defined by

$$\|x\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2.$$

The inner product of $x,\ y$ in $\ell_2(J)$ is therefore defined as the sum $\langle x,\ y\rangle = \sum_{k=0}^{\infty} \langle x_k,\ y_k\rangle$. If the sequence of spaces J is clear from the context we abbreviate to ℓ_2.

One of the most important operators for the derivation of the results in this paper is the unilateral shift operator, which we denote by Z and which is defined as the map

$$Z\colon \ell_2(J_1,\ J_2,\ \ldots) \to \ell_2(J_0,\ J_1,\ J_2,\ \ldots)$$

such that

$$Z(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$$

where $a_i \in J_i$.

II. LTV Systems

Prior to reviewing stability and realization theory results for LTV systems, we present the following notion of block-diagonal operators. This notion is fundamental to the mathematical approach in this paper.

Definition 1: Suppose J and K are sequences of Hilbert spaces. A bounded operator Q mapping $\ell_2(J)$ to $\ell_2(K)$ is ***block-diagonal*** if there exists a sequence of operators $Q_k \in \mathcal{L}(J_k, K_k)$ such that, for all w in $\ell_2(J)$ and z in $\ell_2(K)$, $z = Qw$ implies $z_k = Q_k w_k$. Then Q has the representation

$$\begin{bmatrix} Q_0 & & & 0 \\ & Q_1 & & \\ & & Q_2 & \\ 0 & & & \ddots \end{bmatrix}. \tag{2}$$

If $P_k \in \mathcal{L}(J_k, K_k)$ is a uniformly bounded sequence of operators we say that $P = \mathrm{diag}(P_0, P_1, \ldots)$ is the block-diagonal operator corresponding to P_k and, conversely, given a block-diagonal operator P, the blocks are denoted by P_k, for $k \geq 0$.

A. System Description

Suppose the operator G: $\ell_2 \to \ell_2$ is a linear time-varying discrete time system, described in state-space notation as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k \\ z_k &= C_k x_k + D_k w_k \end{aligned} \tag{3}$$

for $w \in \ell_2$. Here, we allow the state dimension to be time varying, as, for example, in [24] and [29], so that $x_k \in \mathbb{R}^{n_k}$ for each k and, consequently, each matrix A_k has dimension $n_{k+1} \times n_k$. We assume A, B, C and D are bounded sequences of matrices, with $B_k \in \mathbb{R}^{n_{k+1} \times n_u}$, $C_k \in \mathbb{R}^{n_y \times n_k}$ and $D_k \in \mathbb{R}^{n_y \times n_u}$. We assume throughout that the initial condition of the system is $x_0 = 0$. We may then succinctly represent the system in terms of block-diagonal operators, as defined in (2). A number of standard results are now stated in this formalism, and we refer the reader to [30] for further details and proofs of the material in this section.

Using the previously defined notation, clearly A_k, B_k, C_k and D_k in (3) define block-diagonal operators. Recalling that Z is the shift, we can rewrite (3) as

$$\begin{aligned} x &= ZAx + ZBw \\ z &= Cx + Dw \end{aligned} \tag{4}$$

where now this is a system of equations on ℓ_2. Note that, with the notation as used in this paper, the results of [30] hold true for systems with time-varying state dimension, with identical proofs.

Finite-horizon systems:. The results in this paper also apply to time-varying systems on a finite-time interval, with norms defined on $\ell_2[0, T]$, which has finite dimension, rather than on ℓ_2. In this case, one looks at block-diagonal matrices A, B, C, D defined as $A = \mathrm{diag}(A_0, A_1, \ldots, A_T)$, and similarly for B, C and D, with a corresponding finite-dimensional shift operator. Model reduction of finite-horizon systems has interesting applications, for example in construction of low-order models for conservative systems with fluctuation dissipation.

B. Stability

As in the time-invariant case, the system must be stable in order for notions such as balancing and the induced-norm to be well-defined. The standard notion of stability of LTV systems is exponential stability, as follows.

Definition 2: The system G is ***exponentially stable*** if there exist constants $c > 0$ and $0 < \lambda < 1$ such that, for each $j \geq 0$ and any initial condition $x_j \in \mathbb{R}^{n_j}$, the inequality $\|x_k\| \leq c\lambda^{(k-j)}\|x_j\|$ holds for all $k \geq j$.

As is well-known, ℓ_2 stability for the system $x_{k+1} = A_k x_k + v_k$ is equivalent to exponential stability; this is stated in the following proposition. See [37] for systems with time-invariant state-dimension; note that the time-varying dimension case follows immediately.

Proposition 3: Suppose A_k is a bounded sequence of linear maps A_k: $\mathbb{R}^{n_k} \to \mathbb{R}^{n_{k+1}}$. Then, the difference equation $x_{k+1} = A_k x_k$ is exponentially stable if and only if $1 \notin \mathrm{spec}(ZA)$.

If the system is stable, we can rewrite (4) in the linear fractional form

$$G = C(I - ZA)^{-1}ZB + D \tag{5}$$

where $z = Gw$.

Consider the set $\mathcal{T}$ which consists of bounded invertible operators T with bounded inverse, of the form $T = \mathrm{diag}(T_0, T_1, \ldots)$, where each block $T_i \in \mathbb{R}^{n_i \times n_i}$. Given any $T \in \mathcal{T}$ we can define a new realization for the system G according to the following proposition, which is immediate.

Proposition 4: Let

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} Z^*TZ & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

Then

$$C(I - ZA)^{-1}ZB + D = \hat{C}(I - Z\hat{A})^{-1}Z\hat{B} + \hat{D}.$$

That is, these are equivalent realizations for the same system.

C. Generalized Gramians

We now proceed to define a notion of a balanced realization for the system G. Rather than use the standard controllability and observability Gramians that result from solution of Lyapunov equations for this purpose, instead we will make use of *generalized Gramians* [12], [9]. These are solutions of Lyapunov inequalities rather than Lyapunov equations, and have several appealing properties, in particular that the truncation of a balanced system is balanced, as was shown for LTI discrete-time systems in [9]; we make use of this property in the sequel.

We will define the Gramians as block-diagonal operators. Let $\mathcal{X} = \{X > 0;\ X \in \mathcal{T}\}$, the set of strictly positive self-adjoint block-diagonal operators in $\mathcal{T}$.

Lemma 5: The following are equivalent.

i) $1 \notin \operatorname{spec}(ZA)$.

ii) There exists $Y \in \mathcal{X}$ such that

$$AYA^* - Z^*YZ + BB^* < 0. \tag{6}$$

iii) There exists $X \in \mathcal{X}$ such that

$$A^*Z^*XZA - X + C^*C < 0. \tag{7}$$

Proof: First, note that there exists $X \in \mathcal{X}$ such that $ZAXA^*Z^* - X < 0$ if and only if $1 \notin \operatorname{spec}(ZA)$. This is simply the standard result that there exists a quadratic Lyapunov function for A if and only if the system is exponentially stable; one proof is via [30, Th. 11]. The aforementioned inequalities follow immediately from homogeneity and scaling. ■

Note that the generalized Gramians are nonunique and strictly positive, whereas solutions to the corresponding Lyapunov equations (for the time-invariant case) are unique and may not be positive definite. Note that, even in the linear time-invariant case, the use of generalized Gramians can result in tighter error bounds than those obtained through the usual solutions of Lyapunov equations due to the possibility of increasing the multiplicity of the singular values, as was pointed out in [9].

D. Balanced Realizations

We can now define balanced realizations for LTV systems in terms of the generalized Gramians as follows.

Definition 6: A linear time-varying system realization is ***balanced*** if there exist $X, Y \in \mathcal{X}$ satisfying inequalities (7) and (6) such that $X = Y = \Sigma$, where $\Sigma > 0$ is diagonal.

Given a realization for G, we can apply a state transformation according to Proposition 4. Under this state transformation, the generalized Gramians are given by

$$\hat{Y} = TYT^* \quad \hat{X} = (T^*)^{-1}XT^{-1}.$$

Therefore, in order to show that balanced realizations exist, we need to find a transformation T so that $\hat{X}$ and $\hat{Y}$ are diagonal and equal.

Proposition 7: Given positive–definite block-diagonal operators $X \in \mathcal{X}$ and $Y \in \mathcal{X}$, there exists a nonsingular block-diagonal operator $T \in \mathcal{T}$ such that

$$TYT^* = (T^*)^{-1}XT^{-1} = \Sigma$$

where Σ is diagonal and $\Sigma > 0$.

Proof: Since X and Y are block-diagonal and bounded, we can perform a singular value decomposition on $Y^{1/2}XY^{1/2}$ so that

$$Y^{1/2}XY^{1/2} = U\Sigma^2U^*$$

where U is block-diagonal and unitary and Σ is diagonal and positive definite. Then

$$\Sigma^{-(1/2)}U^*Y^{1/2}XY^{1/2}U\Sigma^{-(1/2)} = \Sigma.$$

Set $T^{-1} = Y^{1/2}U\Sigma^{-(1/2)}$, and the result follows.

Corollary 8: Suppose A, B, C, D are bounded block-diagonal operators with $1 \notin \operatorname{spec}(ZA)$. Then, there exists a block-diagonal invertible state transformation $T \in \mathcal{T}$ such that the equivalent realization $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ satisfies

$$\hat{A}\Sigma\hat{A}^* - Z^*\Sigma Z + \hat{B}\hat{B}^* < 0$$
$$\hat{A}^*Z^*\Sigma Z\hat{A} - \Sigma + \hat{C}^*\hat{C} < 0$$

and, hence, is balanced.

The operator Σ is thus block-diagonal, with each of its blocks Σ_k a diagonal matrix. The diagonal elements of Σ_k are called the *Hankel singular values* of the system. They are deeply linked to the model reduction and realization problems; for example, in [19], it is shown that the eigenvalues of X_kY_k are exactly the singular values of a Hankel operator Γ_k^G associated with each time k, and in [24] it is shown that the rank of this Hankel operator determines the number of states required at time k for a realization of the system.

III. Induced Norms for LTV Systems

We now turn to the properties of operators of the form of (5), in particular we consider conditions for evaluating the induced 2-norm.

A. System Function

The representation of the system operator G given by (5) is strongly reminiscent of the frequency domain description for a discrete-time time-invariant system. It is well known that one can construct a map between rational functions on the complex plane and linear time-invariant systems by replacing the shift operator Z by a complex number. The induced norm of the system is then given by the maximum norm of this transfer function over the unit ball in the complex plane.

For linear time-varying systems a similar approach can be taken in computing the induced norm. In this context, we now consider a bounded *sequence* $\lambda_k \in \mathbb{C}$ of complex numbers over which we calculate the maximum norm of an operator-valued function, which leads us to an explicit means for evaluating the induced norm of LTV systems. Essentially, the sequence $\lambda_k \in \mathbb{C}$ of complex numbers serves as our notion of frequency. Given such a sequence, define Λ by

$$\Lambda = \begin{bmatrix} \lambda_0 I_{n_0} & & & 0 \\ & \lambda_1 I_{n_1} & & \\ & & \lambda_2 I_{n_2} & \\ 0 & & & \ddots \end{bmatrix} \tag{8}$$

a block-diagonal operator, where $I_n \in \mathbb{R}^{n\times n}$ is the identity. We can then define the following function.

Definition 9: The ***system function*** $\tilde{G}$ of G is defined by

$$\tilde{G}(\Lambda) = C(I - \Lambda ZA)^{-1}\Lambda ZB + D.$$

The system function plays an instrumental role in our viewpoint, allowing a particularly simple analysis of the induced ℓ_2 norm of LTV systems. The following result is the first step toward a convex condition which we can use to derive the error bound.

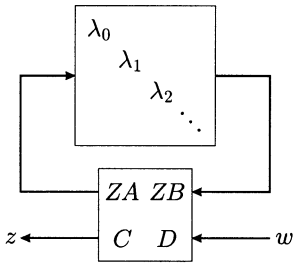

Fig. 1. Linear fractional representation of the system function.

Theorem 10 [30]: Let A, B, C, and D be block-diagonal operators with spatial dimensions as in equation (3). Suppose $1 \notin \operatorname{spec}(ZA)$. Then

$$\|C(I-ZA)^{-1}ZB+D\| = \sup_{|\lambda_k|\le 1} \|\tilde{G}(\Lambda)\|$$

where Λ depends on λ_k as in (8).

Here, the function $\tilde{G}(\lambda)$ both looks and acts very much like the transfer function of an LTI system. Just as for the time-invariant case, the induced norm of the linear time-varying system G is analyzed by computing the maximum norm of an operator-valued function over a complex ball.

The system function $\tilde{G}(\Lambda)$ has the form of an LFT on Λ, which is illustrated by the block-diagram of Fig. 1. Via a generalization of the standard separating hyperplane argument used to derive the Kalman–Yakubovich–Popov lemma, one may derive the following result as a consequence of Theorem 10. This gives us a convex characterization of contractive operators.

Theorem 11: The following conditions are equivalent.

i) $\|C(I-ZA)^{-1}ZB+D\| < 1$ and $1 \notin \operatorname{spec}(ZA)$.

ii) There exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0. \quad (9)$$

iii) There exists $X \in \mathcal{X}$ such that

$$\begin{bmatrix} -X & 0 & A^* & C^* \\ 0 & -I & B^* & D^* \\ A & B & -Z^*X^{-1}Z & 0 \\ C & D & 0 & -I \end{bmatrix} < 0. \quad (10)$$

A proof of the equivalence of parts i) and ii) can be found in [30]; part iii) follows immediately upon applying the Schur complement formula. Formally, part ii) of this result looks the same as the corresponding result for linear time-invariant systems, except that the operators ZA and ZB replace the usual A-matrix and B-matrix, and X is block-diagonal. This formal correspondence between results for linear time-invariant systems and their counterparts in the time-varying case is a general property, and provides a simple and direct means for constructing and understanding the relationship between time-invariant and time-varying systems.

In order to state and prove the main results of this paper, we will need a notation for manipulating partitioned block-diagonal operators. That is the subject of the next section.

B. Partitioned Block-Diagonal Operators

In the following, we make use of permutations of block-diagonal operators, and we use the following notation, taken from [30]. Suppose F, G, R and S are block-diagonal operators, and let A be a *partitioned* operator, each of whose elements is a block-diagonal operator, such as

$$A = \begin{bmatrix} F & G \\ R & S \end{bmatrix} = \begin{bmatrix} F_0 & & & G_0 & & \\ & F_1 & & & G_1 & \\ & & \ddots & & & \ddots \\ R_0 & & & S_0 & & \\ & R_1 & & & S_1 & \\ & & \ddots & & & \ddots \end{bmatrix}.$$

We define

$$[A] = \left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right] = \begin{bmatrix} F_0 & G_0 & & & \\ R_0 & S_0 & & & \\ & & F_1 & G_1 & \\ & & R_1 & S_1 & \\ & & & & \ddots \end{bmatrix}$$

and, clearly, we have

$$\left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right] = P_l \begin{bmatrix} F & G \\ R & S \end{bmatrix} P_r$$

for appropriate permutation operators P_l and P_r. We make use of the natural generalization of this notation to operators A with other block structures than the 2×2 shown previously. Note that the aforementioned notation implies

$$\left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right]_k = \begin{bmatrix} F_k & G_k \\ R_k & S_k \end{bmatrix}$$

and, hence, it is immediately clear that positivity is preserved under permutation, so that $A < \beta I$ if and only if $[A] < \beta I$. Similarly, if A and B have the same block structure, then $[A+B] = [A] + [B]$. Also, for A and C with compatible block structures, we have $[AC] = [A][C]$ and

$$\left[\!\left[\begin{matrix} \left[\!\left[\begin{matrix} P & Q \\ R & S \end{matrix} \right]\!\right] & \left[\!\left[\begin{matrix} T \\ U \end{matrix} \right]\!\right] \\ [V \;\; W] & X \end{matrix} \right]\!\right] = \left[\!\left[\begin{matrix} P & Q & T \\ R & S & U \\ V & W & X \end{matrix} \right]\!\right].$$

In the following, we do not distinguish between shift operators with different spatial dimensions. The spatial dimensions of Z can usually be deduced from context except where stated. This allows us to state the following proposition.

Proposition 12: Suppose F, G, R, and S are block-diagonal operators, so that

$$A = \left[\!\left[\begin{matrix} F & G \\ R & S \end{matrix} \right]\!\right] \quad (11)$$

is a block diagonal operator. Let Z be the shift operator. Then

$$Z^*AZ = \left[\!\left[\begin{matrix} Z^*FZ & Z^*GZ \\ Z^*RZ & Z^*SZ \end{matrix} \right]\!\right].$$

Proof: By definition, (11) means

$$A_k = \begin{bmatrix} F_k & G_k \\ R_k & S_k \end{bmatrix}.$$

Also, for any diagonal operator P, the operator Z^*PZ satisfies $(Z^*PZ)_k = P_{k+1}$, where $(Z^*PZ)_k$ denotes the kth block of the diagonal operator Z^*PZ. Hence

$$\begin{aligned}(Z^*AZ)_k &= A_{k+1} \\ &= \begin{bmatrix} F_{k+1} & G_{k+1} \\ R_{k+1} & S_{k+1} \end{bmatrix} \\ &= \begin{bmatrix} (Z^*FZ)_k & (Z^*GZ)_k \\ (Z^*RZ)_k & (Z^*SZ)_k \end{bmatrix}\end{aligned}$$

which immediately implies the result. ■

IV. MODEL REDUCTION FOR TIME-VARYING SYSTEMS

We now arrive at the main topic of this paper, the development of the "twice the sum of the tail" formulas for the error incurred through balanced truncation of a linear time-varying system. We begin by stating a precise formulation of the problem, and then present a few preliminary lemmas which allow us to prove the error bound in a succinct manner.

Suppose (A, B, C, D) is a balanced realization for the system G, with generalized Gramians X and Y where

$$X = Y = \Sigma > 0$$

and Σ is diagonal. Each block $\Sigma_k \in \mathbb{R}^{n_k \times n_k}$ of Σ is a diagonal matrix; we partition each of these blocks into two sub-blocks $\Gamma_k \in \mathbb{R}^{r_k \times r_k}$ and $\Omega_k \in \mathbb{R}^{(n_k - r_k)\times(n_k - r_k)}$, so that

$$\Sigma = \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix}$$

where Γ and Ω are block-diagonal operators. Here, $0 \le r_k \le n_k$ for all k, where we slightly abuse notation to allow matrices with zero dimensions, to allow for the possibilities that either zero states or all states are truncated at a particular time. The singular values corresponding to the states that will be kept are in Γ, and those that will be truncated are in Ω. We assume without loss of generality that in each block Ω_k the singular values are ordered along the diagonal with the largest first. We can now define the *balanced truncation* G_r of the system G.

Definition 13: Suppose (A, B, C, D) is a balanced realization for the system G. Partition A, B and C according to the partitioning of Σ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad C = [C_1 \quad C_2].$$

The ***balanced truncation*** G_r of G is

$$G_r = C_1(I - ZA_{11})^{-1}ZB_1 + D.$$

Note that the aforementioned partitioning is consistent with the block-diagonal structure of A in that each block A_k is partitioned into submatrices. The resulting balanced-truncation has r_k states at each time k.

Lemma 14: Suppose (A, B, C, D) is a balanced realization for the system G. Then, the corresponding balanced truncation G_r is balanced.

Proof: The linear operator inequality for the generalized controllability Gramian (6) is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^* - Z^* \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} Z + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^* < 0.$$

Applying Proposition 12, we have

$$Z^* \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} Z = \begin{bmatrix} Z^*\Gamma Z & 0 \\ 0 & Z^*\Omega Z \end{bmatrix}$$

and, hence

$$A_{11}\Gamma A_{11}^* + A_{12}\Omega A_{12}^* - Z^*\Gamma Z + B_1 B_1^* < 0$$

which immediately implies that

$$A_{11}\Gamma A_{11}^* - Z^*\Gamma Z + B_1 B_1^* < 0.$$

Similarly for the generalized observability Gramian, we have

$$A_{11}^* Z^*\Gamma Z A_{11} - \Gamma + C_1^* C_1 < 0.$$

■

The following algebra, stated in lemma form for ease of reference, will be used to derive the main error bound.

Lemma 15: Let $X \in \mathcal{X}$ and $Y \in \mathcal{X}$ be self-adjoint block-diagonal operators, with $X > 0$ and $Y > 0$. Then

$$\begin{aligned} AYA^* - Z^*YZ + BB^* &< 0 \\ A^*Z^*XZA - X + C^*C &< 0 \end{aligned}$$

if and only if

$$\begin{bmatrix} -S & K^* \\ K & -Z^*RZ \end{bmatrix} < 0$$

where

$$R = \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Y \end{bmatrix}$$

$$S = \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & X \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & A \\ 0 & 0 & C \\ A & B & 0 \end{bmatrix}.$$

Proof: This result follows from a direct application of the Schur complement formula. ■

Lemma 16: Consider the following partitioning of the operator K:

$$K = \begin{bmatrix} 0 & 0 & 0 & A_{11} & A_{12} \\ 0 & 0 & 0 & A_{21} & A_{22} \\ 0 & 0 & 0 & C_1 & C_2 \\ A_{11} & A_{12} & B_1 & 0 & 0 \\ A_{21} & A_{22} & B_2 & 0 & 0 \end{bmatrix}$$

and a realization for $(1/2)(G - G_r)$ given by

$$M = \begin{bmatrix} A_{11} & 0 & 0 & \frac{1}{\sqrt{2}} B_1 \\ 0 & A_{11} & A_{12} & \frac{1}{\sqrt{2}} B_1 \\ 0 & A_{21} & A_{22} & \frac{1}{\sqrt{2}} B_2 \\ -\frac{1}{\sqrt{2}} C_1 & \frac{1}{\sqrt{2}} C_1 & \frac{1}{\sqrt{2}} C_1 & 0 \end{bmatrix}.$$

Now, define two block diagonal unitary operators as follows:

$$L = \begin{bmatrix} -\frac{1}{\sqrt{2}} I_{r_k} & 0 & 0 & \frac{1}{\sqrt{2}} I_{r_k} & 0 \\ \frac{1}{\sqrt{2}} I_{r_k} & 0 & 0 & \frac{1}{\sqrt{2}} I_{r_k} & 0 \\ 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} & 0 & 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} \\ 0 & 0 & I_{n_y} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} I_{n_k - r_k} & 0 & 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} I_{r_k} & \frac{1}{\sqrt{2}} I_{r_k} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} & 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} \\ 0 & 0 & 0 & I_{n_u} & 0 \\ -\frac{1}{\sqrt{2}} I_{r_k} & \frac{1}{\sqrt{2}} I_{r_k} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} I_{n_k - r_k} & 0 & -\frac{1}{\sqrt{2}} I_{n_k - r_k} \end{bmatrix}.$$

Then, K may be viewed as a dilation of M as shown by the transformation

$$LKP = \begin{bmatrix} A_{11} & 0 & 0 & \frac{1}{\sqrt{2}} B_1 & -A_{12} \\ 0 & A_{11} & A_{12} & \frac{1}{\sqrt{2}} B_1 & 0 \\ 0 & A_{21} & A_{22} & \frac{1}{\sqrt{2}} B_2 & 0 \\ -\frac{1}{\sqrt{2}} C_1 & \frac{1}{\sqrt{2}} C_1 & \frac{1}{\sqrt{2}} C_1 & 0 & -\frac{1}{\sqrt{2}} C_2 \\ A_{21} & 0 & 0 & \frac{1}{\sqrt{2}} B_2 & A_{22} \end{bmatrix} = \begin{bmatrix} M & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

where the last equation defines N_{12}, N_{21}, and N_{22}.

Notice that in the previous definitions of L and P we have abused notation slightly, so that I_{r_k} means a block diagonal operator with the kth block equal to I_{r_k}.

A. Main Results

We first consider the special case of $\Omega_k = I_{n_k - r_k}$ for all k, that is where all truncated singular values are equal to one. Note that, for some k, we may have $n_k = r_k$, in which case no states are truncated.

Theorem 17: Suppose (A, B, C, D) is a balanced realization for the system G, with generalized Gramians X and Y where $X = Y = \Sigma$. Suppose that Σ has the form

$$\Sigma = \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix}$$

and $\Omega_k = I_{n_k - r_k}$. Let G_r be the balanced truncation G. Then, the error between G and G_r satisfies the induced-norm bound

$$\|G - G_r\| < 2$$

and the reduced system G_r is balanced.

Proof: Since X and Y satisfy the Lyapunov inequalities, Lemma 15 implies

$$\begin{bmatrix} -S & K^* \\ K & -Z^*RZ \end{bmatrix} < 0.$$

Hence, since P and L are unitary

$$\begin{bmatrix} P^* & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} -S & K^* \\ K & -Z^*RZ \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & L^* \end{bmatrix} < 0 \qquad (12)$$

which is equivalent to

$$\begin{bmatrix} -P^*SP & P^*K^*L^* \\ LKP & -Z^*LRL^*Z \end{bmatrix} < 0.$$

Algebraic manipulations lead to

$$P^*SP = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \quad LRL^* = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

where

$$S_1 = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1} + \Gamma) & \frac{1}{2}(\Gamma^{-1} - \Gamma) \\ \frac{1}{2}(\Gamma^{-1} - \Gamma) & \frac{1}{2}(\Gamma^{-1} + \Gamma) \end{bmatrix}$$

$$S_2 = \begin{bmatrix} \frac{1}{2}(\Omega^{-1} + \Omega) & 0 & \frac{1}{2}(\Omega^{-1} - \Omega) \\ 0 & I & 0 \\ \frac{1}{2}(\Omega^{-1} - \Omega) & 0 & \frac{1}{2}(\Omega^{-1} + \Omega) \end{bmatrix}$$

and

$$R_1 = \begin{bmatrix} \frac{1}{2}(\Gamma^{-1} + \Gamma) & \frac{1}{2}(\Gamma - \Gamma^{-1}) \\ \frac{1}{2}(\Gamma - \Gamma^{-1}) & \frac{1}{2}(\Gamma^{-1} + \Gamma) \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \frac{1}{2}(\Omega^{-1} + \Omega) & 0 & \frac{1}{2}(\Omega - \Omega^{-1}) \\ 0 & I & 0 \\ \frac{1}{2}(\Omega - \Omega^{-1}) & 0 & \frac{1}{2}(\Omega^{-1} + \Omega) \end{bmatrix}.$$

Note that in the aforementioned expressions for R_2 and S_2, if $n_k = r_k$ then the first and third rows and columns are not present, and the corresponding operator inequalities remain valid. Let

$$V = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix}$$

where the partitioning corresponds to LRL^*. Since $\Omega = I$, then

$$(VP^*SPV^*)^{-1} = VLRL^*V^*. \tag{13}$$

The operator $\overline{X}$, defined by

$$\overline{X} = VP^*SPV^*$$

will serve the role of X in (10) in Theorem 11, which we will apply to M in order to show contractiveness of $(1/2)(G - G_r)$. To see this, define $\overline{W}$ by

$$\overline{W} = \left[\!\!\left[\begin{matrix} \overline{X} & 0 \\ 0 & I_{n_u} \end{matrix}\right]\!\!\right]$$

using which we can conclude that (12) is equivalent to

$$\begin{bmatrix} -\overline{W} & 0 & M^* & N_{21}^* \\ 0 & -I & N_{12}^* & N_{22}^* \\ M & N_{12} & -Z^*\overline{W}^{-1}Z & 0 \\ N_{21} & N_{22} & 0 & -I \end{bmatrix} < 0$$

and, therefore

$$\begin{bmatrix} -\overline{W} & M^* \\ M & -Z^*\overline{W}^{-1}Z \end{bmatrix} < 0.$$

Theorem 11 then implies that the error-map $(1/2)(G - G_r)$ is contractive. ■

The preceding result provides an error bound for the special case where all of the truncated singular values are equal to one. We now generalize this to the first major result of this section.

Theorem 18: Suppose (A, B, C, D) is a balanced realization for the system G, with generalized Gramians X and Y where $X = Y = \Sigma$. Suppose that Σ has the form

$$\Sigma = \left[\!\!\left[\begin{matrix} \Gamma & 0 \\ 0 & \Omega \end{matrix}\right]\!\!\right]$$

where for all k, $\Gamma_k \in \mathbb{R}^{r_k \times r_k}$ and $\Omega_k \in \mathbb{R}^{n_k - r_k \times n_k - r_k}$, and Ω_k satisfies

$$\Omega_k = \operatorname{diag}(\sigma_1 I_{d_{1,k}}, \sigma_2 I_{d_{2,k}}, \ldots, \sigma_p I_{d_{p,k}})$$

where

$$\sum_{i=1}^{p} d_{i,k} = n_k - r_k.$$

Let G_r be the balanced truncation of G. Then

$$\|G - G_r\| < 2\sum_{i=1}^{p} \sigma_i. \tag{14}$$

Proof: The proof follows from scaling and repeated application of the previous theorem. ■

This theorem considers a special case of the general linear time-varying model reduction problem; that in which the singular values being truncated are constant over time. Note however that their *multiplicities* may vary with time, and may indeed be zero at some times; that is, for some i, k we may have $d_{i,k} = 0$. Here, p denotes the maximum number of distinct singular values truncated at any time k. Since we may choose which singular values to truncate, we can apply the previous theorem directly even when the sequence of truncated singular values is not constant over time; for example, for the sequence of Ω_k given by

$$\Omega = \operatorname{diag}\left(\begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 5 & \\ & 1 \end{bmatrix} [3] \begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix}, \ldots\right)$$

the corresponding error bound derived by directly applying the theorem is 18. However, with the sequence

$$\Omega = \operatorname{diag}\left(\begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 5 & & \\ & 2 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 5 & \\ & 1 \end{bmatrix} [3] \begin{bmatrix} 5 & & \\ & 3 & \\ & & 1 \end{bmatrix}, \ldots\right)$$

two applications of the above theorem are required, resulting in an error bound of 22. When singular values vary over time, we may always apply the aforementioned theorem as many times as necessary to remove all distinct singular values at all times, resulting in an error bound equal to twice the sum of the distinct singular values. This is stated in the following theorem.

Theorem 19: Suppose (A, B, C, D) is a balanced realization for the system G, with generalized Gramians X and Y where $X = Y = \Sigma$. Suppose that Σ has the form

$$\Sigma = \left[\!\!\left[\begin{matrix} \Gamma & 0 \\ 0 & \Omega \end{matrix}\right]\!\!\right]$$

where for all k, $\Gamma_k \in \mathbb{R}^{r_k \times r_k}$ and $\Omega_k \in \mathbb{R}^{n_k - r_k \times n_k - r_k}$, and Ω_k satisfies

$$\Omega_k = \operatorname{diag}(\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{n_k - r_k, k}).$$

Let G_r be the balanced truncation of G. Then

$$\|G - G_r\| < 2\sum \left\{ j \middle| j = \sigma_{i,k}, \text{ for some } i, k \right\}.$$

Proof: The proof follows from repeated application of the previous theorem. ■

Periodic systems: Note that, for q-periodic systems there are exactly q Gramian matrices, and this result immediately implies the balanced truncation error is bounded by twice the sum of their distinct singular values, as first shown in [27].

B. Distinct Singular Values

If there are infinitely many distinct singular values which do not decay sufficiently fast, then the sum in Theorem 19 may not be finite, in which case one cannot conclude any error-bound. To avoid this summation of singular values over time, one would like to generalize Theorem 17 to the case when $\Omega \neq I$. However, in this case

$$\tfrac{1}{2}(\Omega_k^{-1} + \Omega_k) \neq \left(\tfrac{1}{2}(\Omega_k^{-1} + \Omega_k)\right)^{-1}$$

which then implies that (13) no longer holds. For the above proof technique to hold, therefore, the truncated elements of the generalized Gramians with nonzero dimension must be constant over time. In fact, such generalized Gramians, which allow only one application of Theorem 18, always exist, although it may be difficult to compute suitable choices. One way to show this is via [19, Th. 3], which shows that $\|A\| < 1$ for a balanced LTV system. Hence, there always exist block-diagonal operators such that

$$A^* Z^* X Z A - X < 0$$
$$A Y A^* - Z^* Y Z < 0.$$

One choice satisfying these conditions is the trivial one, $X = Y = I$. Hence, by scaling, we can choose the Gramians to be equal. Of course, typically these will not be a *good* choice, since they will often give poor bounds for the error in balanced truncation; however, they do provide initial evidence that such twice-the-sum-of-the-tail bounds always exist. In the next section, we will see that there always exist constant Gramians that achieve "good" error bounds, in a sense which parallels that for time-invariant systems.

V. Necessary and Sufficient Model Reduction Condition

We have seen in the previous section that the error between a system G and its balanced truncation G_r is bounded by formula (14). This section considers the following problem; given $\gamma > 0$, what are the necessary and sufficient conditions on G such that there exists a G_r of given state dimensions satisfying

$$\|G - G_r\| < \gamma.$$

We briefly outline a result which provides an answer to this question, although this condition does not yield convex computational solutions. However, an important consequence of this is to show that truncated generalized singular values which are constant over time, again allowing for varying or zero multiplicities, is *not* an exceptional case, and that such balanced realizations always exist.

A. Robust Synthesis and Model Reduction

We begin by noting that this particular model reduction problem can be viewed equivalently as a reduced-order robust synthesis problem for time-varying systems, defined as follows. Let P be a linear time-varying system partitioned as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

Then, the reduced-order synthesis problem is, given $\gamma > 0$, find K such that $\|\mathcal{F}_l(P, K)\| < \gamma$. Here, $\mathcal{F}_l(P, K)$ is the lower LFT defined by

$$\mathcal{F}_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}.$$

The interconnection of P and K is illustrated in Fig. 2. If we set

$$P = \begin{bmatrix} G & -I \\ I & 0 \end{bmatrix} \tag{15}$$

then

$$\mathcal{F}_l(P, K) = G - K$$

which is the form needed for the model reduction problem. This approach of converting a model reduction problem to a synthesis problem is standard; see, for example, [12].

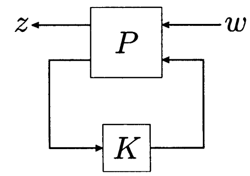

Fig. 2. Model reduction as an LFT synthesis.

We first define the system P and the controller K by state-space equations. P is given by

$$\begin{aligned} x_{k+1} &= A_{1k} x_k + B_{1k} w_k + B_{2k} u_k \qquad x_0 = 0 \\ z_k &= C_{1k} x_k + D_{11k} w_k + D_{12k} u_k \\ y_k &= C_{2k} x_k + D_{21k} w_k \end{aligned} \tag{16}$$

where $x_k \in \mathbb{R}^{n_k}$, $w_k \in \mathbb{R}^{n_w}$, $u_k \in \mathbb{R}^{n_u}$, $z_k \in \mathbb{R}^{n_z}$, and $y_k \in \mathbb{R}^{n_y}$, and the matrices A_1, B_i, C_i, D_{ij} are assumed to be uniformly bounded functions of time. The reduced-system K, or controller in this formulation, has the form

$$\begin{aligned} x_{k+1}^K &= A_k^K x_k^K + B_k^K y_k \\ u_k &= C_k^K x_k^K + D_k^K y_k \end{aligned} \tag{17}$$

where $x_k^K \in \mathbb{R}^{r_k}$. This interconnection is well-posed, since the direct feed-through term of P is $D_{22} = 0$. Then, the generator of the closed-loop system is given by

$$A^L = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A^K & B^K \\ C^K & D^K \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}.$$

The conditions for time-varying synthesis are given by the following theorem.

Theorem 20: Given the previous system, there exists a controller K with time-varying state-dimensions r_k such that $1 \notin \operatorname{spec}(ZA^L)$ and the closed-loop performance inequality $\|w \mapsto z\|_{\ell_2 \to \ell_2} < 1$ holds if and only if there exist block-diagonal operators $Y \in \mathcal{X}$ and $X \in \mathcal{X}$ satisfying the inequalities

$$N^* \begin{bmatrix} A_1 Y A_1^* - Z^* Y Z & A_1 Y C_1^* & B_1 \\ C_1 Y A_1^* & C_1 Y C_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} N < 0$$

$$M^* \begin{bmatrix} A_1^* Z^* X Z A_1 - X & A_1^* Z^* X Z B_1 & C_1^* \\ B_1^* Z^* X Z A_1 & B_1^* Z^* X Z B_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} M < 0$$

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} \geq 0$$

$$\operatorname{rank} \begin{bmatrix} Y_k & I \\ I & X_k \end{bmatrix} \leq n_k + r_k$$

where the operators N_Y, M_X satisfy

$$\operatorname{image} N_Y = \ker \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \quad N_Y^* N_Y = I$$
$$\operatorname{image} M_X = \ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \quad M_X^* M_X = I$$

and

$$N = \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} \quad M = \begin{bmatrix} M_X & 0 \\ 0 & I \end{bmatrix}.$$

The proof of this result is at the end of this section. Relating the realization for P as defined in (16) to (15), we now have

$$\begin{aligned} A_1 &= A \\ B_1 &= B \quad B_2 = 0 \\ C_1 &= C \quad C_2 = 0 \\ D_{11} &= D \quad D_{12} = -I \\ D_{21} &= I \quad D_{22} = 0. \end{aligned} \tag{18}$$

Substituting (18) into Theorem 20 leads immediately to the following.

Theorem 21: There exists a reduced model G_r of G, with state dimensions $r_k \leq n_k$ and $\|G-G_r\| < 1$, if and only if there exist block-diagonal operators $Y \in \mathcal{X}$ and $X \in \mathcal{X}$ satisfying

$$\begin{bmatrix} AYA^* - Z^*YZ & B \\ B^* & -I \end{bmatrix} < 0$$
$$\begin{bmatrix} A^*ZXZA - X & C^* \\ C & -I \end{bmatrix} < 0$$
$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} \geq 0$$
$$\operatorname{rank}(Y_k - X_k^{-1}) = r_k, \qquad \text{for all } k.$$

Further application of the Schur complement formula, together with scaling, leads to the following.

Theorem 22: There exists a reduced model G_r of G, with state dimensions $r_k \leq n_k$ and $\|G-G_r\| < \gamma$ if and only if there exist block-diagonal operators $Y \in \mathcal{X}$ and $X \in \mathcal{X}$ satisfying

$$AYA^* - Z^*YZ + BB^* < 0$$
$$A^*ZXZA - X + C^*C < 0$$

such that, for all k

$$\lambda_{\min}(X_k Y_k) = \gamma^2$$

with multiplicity $n_k - r_k$.

This brings us to the necessary condition we required, in terms of the generalized Gramians X and Y. Note that the eigenvalues of $X_k Y_k$ are invariant under change of realization coordinates and, hence, in particular are unchanged by balancing. Thus, we can conclude that, if there exists a reduced-order model which achieves an error-bound of γ, there *will always exist Gramians with $n_k - r_k$ repeated singular values equal to* γ, and hence Theorem 18 can be applied. Thus the balanced truncation procedure will give an error-bound of 2γ. This exactly parallels the well-known results in the time-invariant case.

1) Proof of Theorem 20: Theorem 20 is a slight extension of [30, Th. 19], which provides a proof of the necessary and sufficient conditions for full-order control synthesis for LTV systems. The proof also holds for the reduced-order case, apart from Lemma 18 in that paper, which must be modified to the following.

Lemma 23: Suppose $R > 0$ and $S > 0$ are block-diagonal operators with entries $R_k, S_k \in \mathbb{R}^{n_k \times n_k}$. Then, there exists $X_{2_k} \in \mathbb{R}^{n_k \times r_k}$ and $X_{3_k} \in \mathbb{R}^{r_k \times r_k}$ with $X_{3_k} > 0$ such that

$$\begin{bmatrix} R & X_2 \\ X_2^* & X_3 \end{bmatrix} > 0$$

and

$$\begin{bmatrix} R_k & X_{2_k} \\ X_{2_k}^* & X_{3_k} \end{bmatrix}^{-1} = \begin{bmatrix} S_k & Y_{2_k} \\ Y_{2_k}^* & Y_{3_k} \end{bmatrix}, \qquad \text{for all } k$$

iff

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0$$

and

$$\operatorname{rank} \begin{bmatrix} R_k & I \\ I & S_k \end{bmatrix} \leq n_k + r_k \quad \text{for all } k.$$

The proof of this result is essentially identical to the matrix version found in [12], and is, therefore, omitted.

VI. Conclusion

In this paper, we have derived error bounds for balanced truncation of linear time-varying systems. The main tools used are an operator representation for time-varying systems together with representation of model reduction and realization as a linear-fractional transformation. The mathematical links here are very interesting, touching upon the theories of realization, identification, and Hankel operators. There are also important applications of linear time-varying control, such as control of nonlinear systems along trajectories.

We have made use of an extended notion of generalized Gramians for time-varying systems, expressed in terms of linear matrix inequalities, allowing standard, but infinite dimensional, convex programming techniques to be used in their construction. The error-bounds take the form of "twice-the-sum-of-the-tail" formula which are well-understood for linear time-invariant systems, and the bounds in the time-varying case reduce to those of the time-invariant case when applied to time-invariant systems

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