EE464: Convexity Review

Linear programming

Semidefinite programming

Linear programming

Linear programming

a linear program in *standard primal form*

$$\begin{array}{rl} \text{minimize} & c^{\mathsf{T}} x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- many other forms, *e.g.*, using slack variables or splitting variables
- feasible set is intersection of affine subspace with nonnegative orthant
- intersection of two convex sets, hence convex
- a *polyhedron* is the intersection of finitely many closed halfspaces
- a bounded polyhedron is called a *polytope*

Dual LP

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

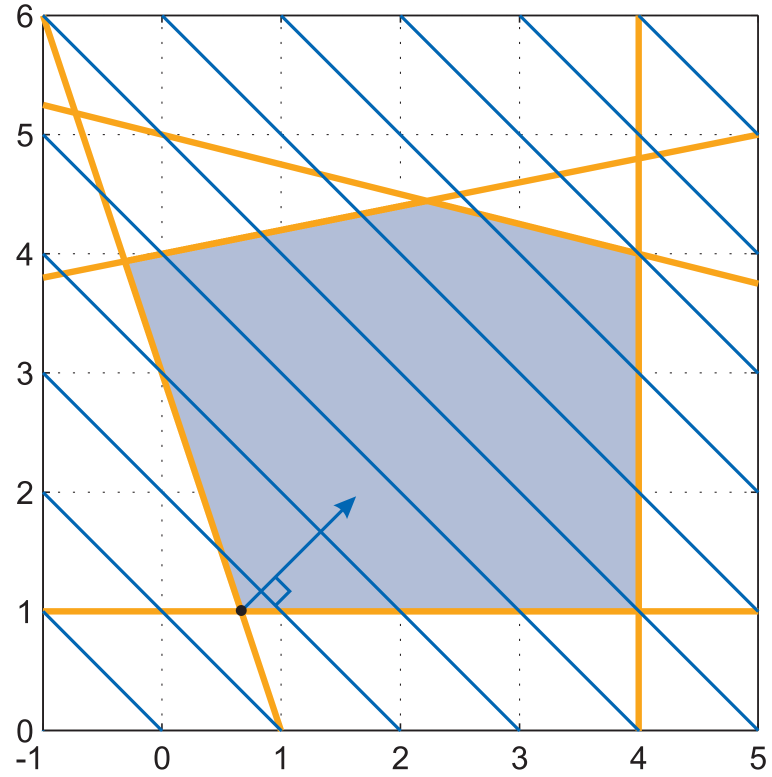
- again, optimizing a linear function over a polyhedron
- several direct relationships to the primal problem

Complexity of LP

- $A \in \mathbb{R}^{m \times n}$
- L is bit-length of the input
- Klee and Minty (1972) example; simplex algorithm takes 2^n steps
- Khachiyan (1979) gave (impractical) ellipsoid algorithm taking $O(n^6 L^2)$
- *i.e.*, polynomial in the Turing model, called *weakly polynomial time*
- Karmarkar (1984) gave practical interior-point method, also weakly polynomial
- unknown if there is a strongly polynomial algorithm, *i.e.*, one polynomial in n, m

Example: Linear program

$$\begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & 3x_1 + x_2 \geq 3 \\ & x_2 \geq 1 \\ & x_1 \leq 4 \\ & -x_1 + 5x_2 \leq 20 \\ & x_1 + 4x_2 \leq 20 \end{array}$$



Properties of linear programs

- *feasible set is a polyhedron*, hence has finitely many extreme points and extreme rays

- every polyhedron P has the form

$$P = \mathbf{conv}(u_1, \ldots, u_r) + \mathbf{cone}(v_1, \ldots, v_s)$$

  where $u_1, \ldots, u_r$ are the vertices and $v_1, \ldots, v_r$ are the extreme rays

- the vertices provide an alternative representation of a polytope

- the representation of the feasible set can affect the practical computational cost of solving a linear program

Properties of linear programs

- if optimal value is achieved, then it is *achieved at an extreme point*
- for a polyhedron, extreme points are *rational* functions of A, b, c

Properties of linear programs

- *weak duality:* if x, y are both feasible points, then

$$c^T x - b^T y \geq 0$$

because $c^T x - b^T y = x^T(c - A^T y) \geq 0$

- *strong duality:* the primal is feasible iff the dual is feasible. If feasible, they have the same optimal value

- *complementary slackness:* if x, y feasible, then they are optimal iff

$$x_i(c - A^T y)_i = 0 \qquad \text{for all } i$$

(follows from strong duality and above inequality)

Semidefinite programming

Positive definite matrices

- $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$ denote the sets of $n \times n$ symmetric, positive semidefinite, and positive definite matrices

- $S \subset \mathbb{R}^m$ is called a *spectrahedron* if it has the form

$$S = \left\{ x \in \mathbb{R}^m \mid A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0 \right\}$$

  where $A_0, \ldots, A_m$ are symmetric matrices

- above inequality is called a *linear matrix inequality*

- a spectrahedron is closed and convex, since it is the intersection of an affine subspace and the positive semidefinite cone

Positive definite matrices

- some authors define a spectrahedron as a set of matrices

$$\left\{A_0 + \sum_{i=1}^{m} A_i x_i \,\middle|\, x \in S\right\}$$

this set is affinely equivalent to S if the A_i are linearly independent

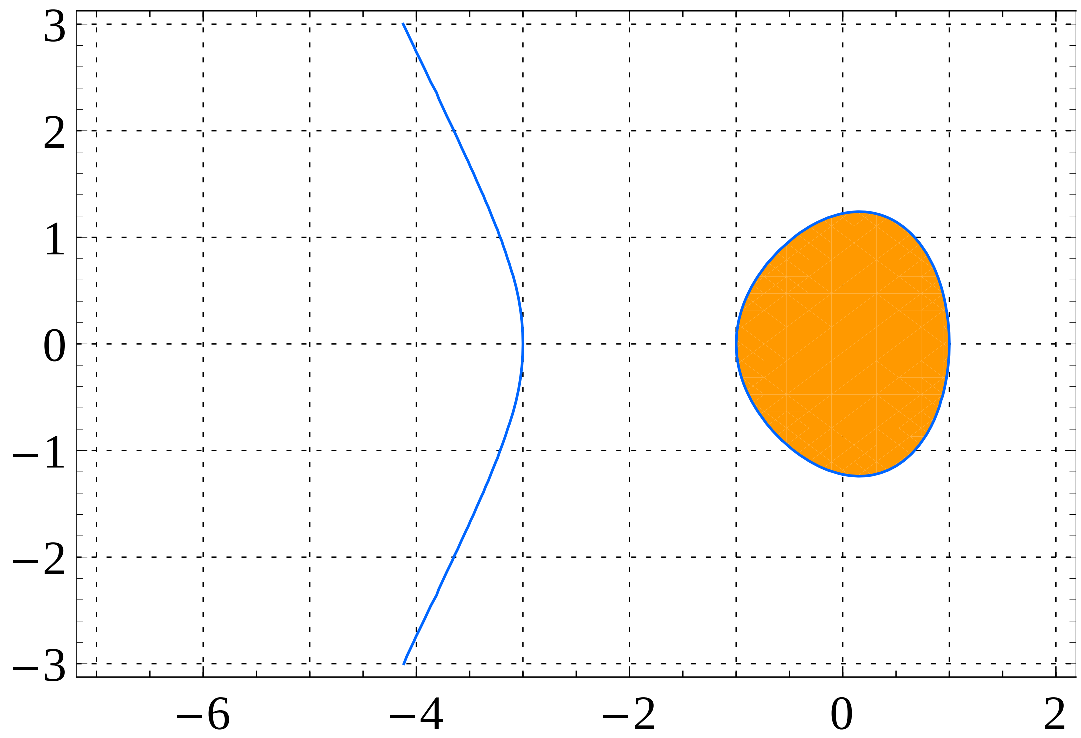
- $S \subset \mathbb{R}^m$ is called a *projected spectrahedron* if it has the form

$$S = \left\{x \in \mathbb{R}^m \,\middle|\, \exists y \; A_0 + \sum_{i=1}^{m} A_i x_i + \sum_{i=1}^{p} B_i y_i \succeq 0\right\}$$

where $A_0, \ldots, A_m, B_1, \ldots, B_p$ are symmetric matrices

Example: Spectrahedron

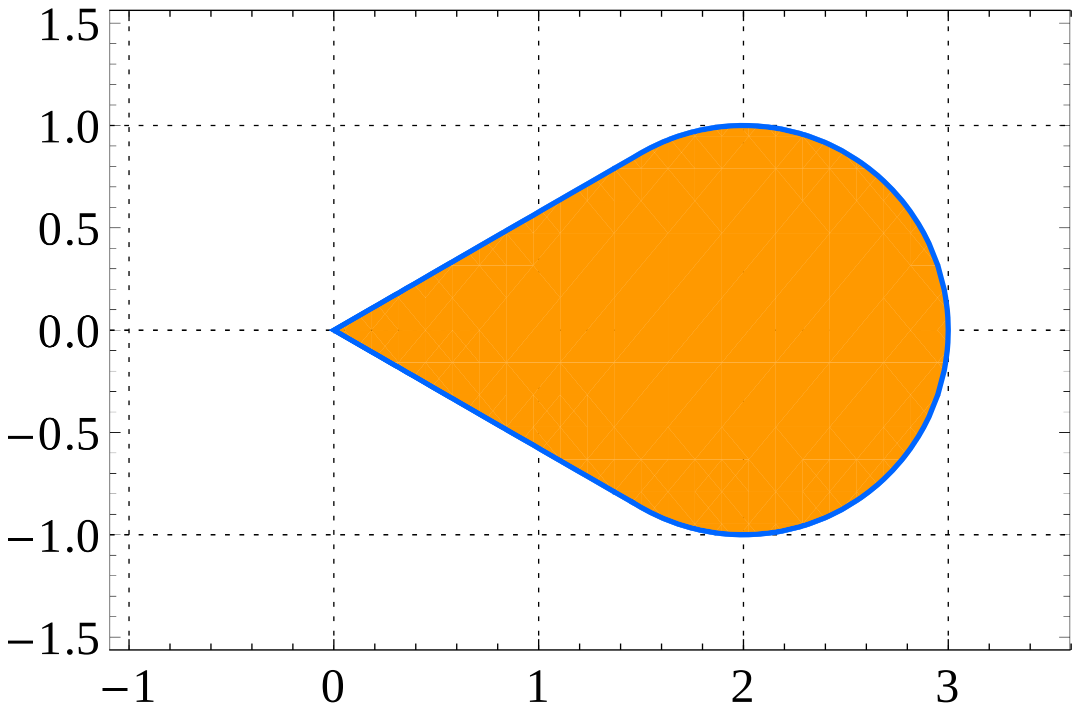
$$\left\{ (x, y) \in \mathbb{R}^2 \mid \begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix} \succeq 0 \right\}$$



notice that the determinant $3 + x - 3x^2 - x^3 - 2y^2$ vanishes on the boundary

Example: Projected spectrahedron

$$\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \exists z \in \mathbb{R},\ \begin{bmatrix} z+y & 2z-x \\ 2z-x & z-y \end{bmatrix} \geq 0,\ 0 \leq z \leq 1 \right\}$$



this set is the convex hull of $(x-2)^2 + y^2 \leq 1$ and the origin

it is *not* a spectrahedron

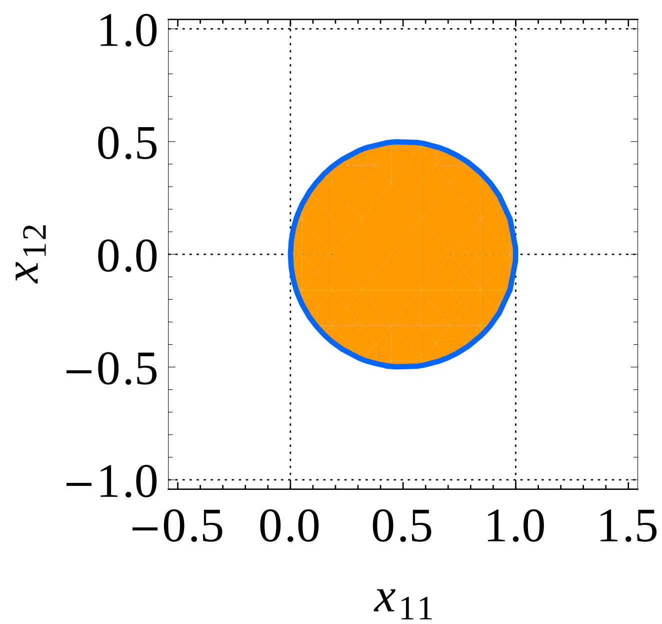
Semidefinite programming

$$
\begin{array}{rll}
\text{minimize} & \langle C, X \rangle & \\
\text{subject to} & \langle A_i, X \rangle = b_i & \text{for all } i = 1, \ldots, m \\
& X \succeq 0 &
\end{array}
$$

- variables are $X \in \mathbb{S}^n$
- $C, A_i \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$
- formally similar to LP
- convex, since spectrahedron is convex

Example: Semidefinite programming

$$\begin{array}{ll} \text{minimize} & 2x_{11} + 2x_{12} \\ \text{subject to} & x_{11} + x_{22} = 1 \\ & \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq 0 \end{array}$$



- standard form, with

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad b_1 = 1$$

- feasible set is not polyhedral
- optimal is not rational: $X = \frac{1}{4}\begin{bmatrix} 2-\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 2+\sqrt{2} \end{bmatrix}$

Semidefinite programming

there may not exist an optimal solution

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} x & 1 \\ 1 & t \end{bmatrix} \succeq 0 \end{array}$$

Dual SDP

$$\begin{array}{ll} \text{maximize} & b^\mathsf{T} y \\ \text{subject to} & \sum_{i=1}^{m} A_i y_i \preceq C \end{array}$$

- *weak duality*; if X, y are feasible, then $\langle C, X \rangle - b^\mathsf{T} y \geq 0$

Strong duality

define the dual value

$$d^\star = \sup\{b^\mathsf{T} y \mid \sum_{i=1}^{m} A_i y_i \preceq C\}$$

Strong duality

if the dual is strictly feasible, *i.e.*, there exists y such that

$$\sum_{i=1}^{m} A_i y_i \prec C$$

and the dual problem is bounded, *i.e.*, $d^\star$ is finite, then

- **primal feasibility**: there exists $X \succeq 0$ such that

$$\langle A_i, X \rangle = b_i \qquad \text{for all } i = 1, \ldots, m$$

- **optimality**: that X is optimal

$$\langle C, X \rangle = d^\star$$

Semialgebraic sets

the feasible set of an SDP has the form

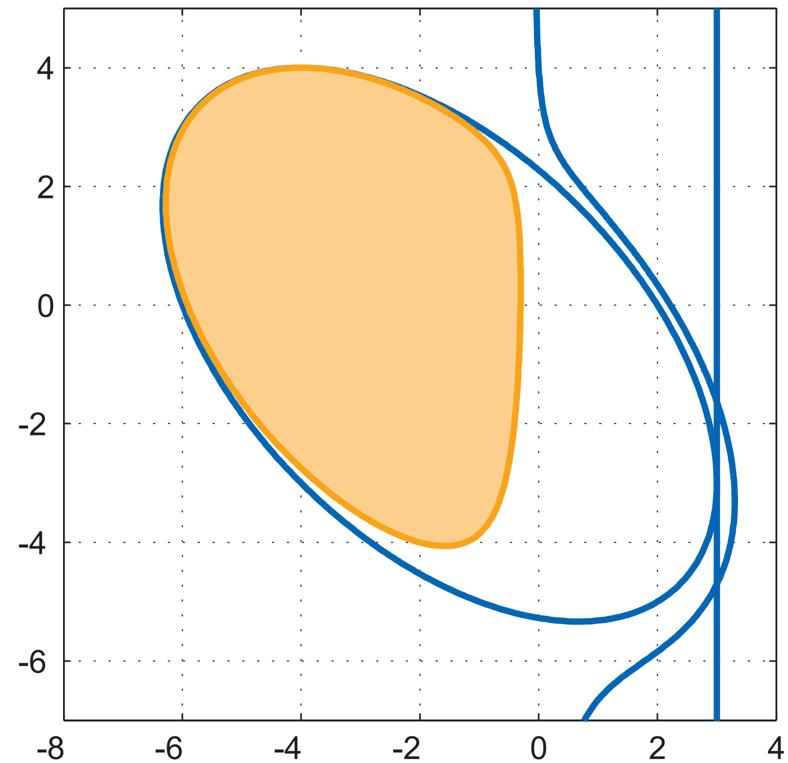
$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \right\}$$

- $f_1, \ldots, f_m$ is are polynomials
- called a *basic closed semialgebraic set* defined by
- because a matrix $A \succ 0$ if and only if

  $$\det(A_k) > 0 \text{ for } k = 1, \ldots, n$$

  where A_k is the submatrix of A consisting of the first k rows and columns

Example: semialgebraic set



$$0 \prec \begin{bmatrix} 3-x_1 & -(x_1+x_2) & 1 \\ -(x_1+x_2) & 4-x_2 & 0 \\ 1 & 0 & -x_1 \end{bmatrix}$$

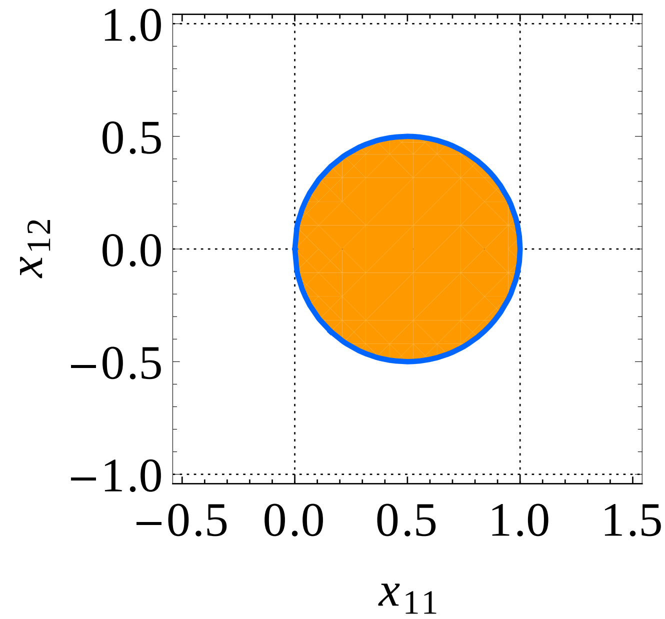
is equivalent to the polynomial inequalities

$$0 < 3 - x_1$$

$$0 < (3-x_1)(4-x_2) - (x_1+x_2)^2$$

$$0 < -x_1((3-x_1)(4-x_2) - (x_1+x_2)^2) - (4-x_2)$$

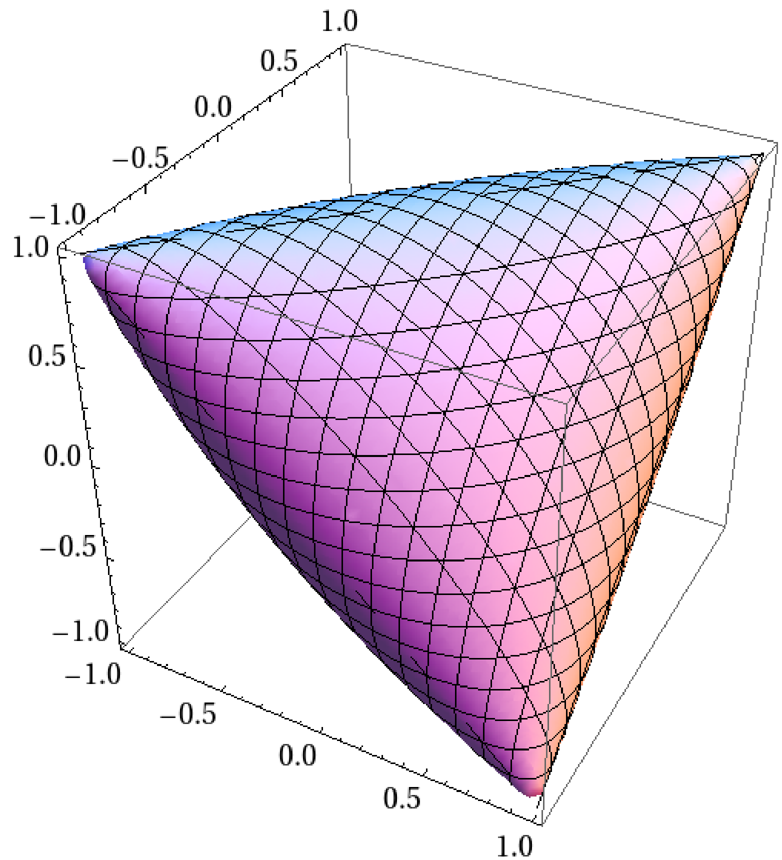
Spectraplex



$$\mathcal{O}_n = \{X \in \mathbb{S}^n \mid X \succeq 0, \mathbf{tr}\, X = 1\}$$

- called the *spectraplex* or *free spectrahedron*
- compact
- extreme points are rank 1, of the form $X = xx^\mathsf{T}$ with $\|x\| = 1$
- $\mathcal{O}_2$ is isomorphic to the closed unit disk in $\mathbb{R}^2$

Elliptope



- $\mathcal{E}_n = \{X \in \mathbb{S}^n \mid X \succeq 0, X_{ii} = 1 \text{ for } i = 1, \ldots, n\}$
- compact
- important in combinatorial optimization

Operator norm

$$\begin{array}{ll} \text{maximize} & 2\,\mathbf{tr}\,A^{\mathsf{T}} X_{12} \\ \text{subject to} & \mathbf{tr} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix} = 1 \\ & X \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} tI & A \\ A^{\mathsf{T}} & tI \end{bmatrix} \succeq 0 \end{array}$$

- dual pair of SDPs
- value equal to the operator norm $\|A\|$
- figure shows unit ball for $A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$