EE464 Resultants
**companion matrix**

write \( p = x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0 \) in terms of its roots \( x_1, \ldots, x_n \):

\[
p(x) = \prod_{k=1}^{n} (x - x_k)
\]

define the \( n \times n \) companion matrix

\[
C_p = \begin{bmatrix}
0 & 0 & \ldots & 0 & -p_0 \\
1 & 0 & \ldots & 0 & -p_1 \\
0 & 1 & \ldots & 0 & -p_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & -p_{n-1}
\end{bmatrix}
\]

the characteristic polynomial of \( C_p \) is \( p \)

\[
\det(xI - C_p) = p
\]
eigenvectors of the companion matrix

define the Vandermonde matrix

\[
V = \begin{bmatrix}
1 & x_1 & \ldots & x_1^{n-1} \\
1 & x_2 & \ldots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \ldots & x_n^{n-1}
\end{bmatrix}
\]

\[VC_p = \text{diag}(x_1, \ldots, x_n)V\]

\[V\text{ is nonsingular iff the } x_i \text{ are distinct}\]

\[\text{columns of } V^{-1} \text{ are coefficients of Lagrange polynomials } L_j(x_i) = \delta_{ij}\]

because \[\begin{bmatrix}1 & x_1 & \ldots & x_1^{n-1}\end{bmatrix}V^{-1} = e_1^T V V^{-1} = e_1^T\]
\[ p = (x - 1)(x - 2)(x - 5) \]

\[ C_p = \begin{bmatrix} 0 & 0 & 10 \\ 1 & 0 & -17 \\ 0 & 1 & 7 \end{bmatrix} \]

\[ V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{bmatrix} \quad V^{-1} = \frac{1}{12} \begin{bmatrix} 30 & -20 & 2 \\ -21 & 24 & -3 \\ 3 & -4 & 1 \end{bmatrix} \]

\[ L_1(x) = \frac{(30 - 21x + 3x^2)}{12} = \frac{(x - 2)(x - 5)}{4} \]
trace of the companion matrix

for any $A \in \mathbb{C}^{n \times n}$ we have

$$\text{trace } A = \sum_{i=1}^{n} \lambda_i(A) \quad \lambda_i(A^k) = \lambda_i(A)^k$$

hence trace of powers of companion matrix gives sum of root powers

$$\text{trace } C_p^k = \sum_{i=1}^{n} x_i^k$$
symmetric functions of roots

if \( q = q_0 + q_1 x + \ldots q_m x^m \) then

\[
\sum_{i=1}^{n} q(x_i) = \text{trace} \ q(C_p)
\]

because

\[
\sum_{i=1}^{n} q(x_i) = \sum_{i=1}^{n} \sum_{j=0}^{m} q_j x_i^j = \sum_{j=0}^{m} q_j \text{trace} \ C_p^j = \text{trace} \sum_{j=0}^{m} q_j C_p^j = \text{trace} \ q(C_P)
\]
Hermite form

given polynomials \( p \) and \( q \), the Hermite form is the symmetric matrix

\[
H_q(p) = V^T \text{diag}(q(x_1), \ldots, q(x_n))V
\]

- with \( q(x) = 1 \), we have

\[
H_1(p) = V^T V = \begin{bmatrix}
    s_0 & s_1 & \cdots & s_{n-1} \\
    s_1 & s_2 & \cdots & s_n \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{n-1} & s_n & \cdots & s_{2n-2}
\end{bmatrix}
\]

- can compute using \( s_k = \text{trace} C^k_p \)

- signature of \( M \) is the number of positive eigenvalues minus the number of negative eigenvalues

- theorem: signature of \( H_1(p) \) = the number of real roots of \( p \).

\( \text{rank} H_1(p) \) = the number of distinct complex roots of \( p \)
Hermite form

- $p = x^2 + 2x^2 + 3x + 4$

- $H_1(p) = \begin{bmatrix} 3 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & 18 \end{bmatrix}$

- $H_1(p)$ has one negative and two positive eigenvalues

- hence $p$ has three simple roots, one of them is real
scalar polynomials

when do two polynomials $f, g \in \mathbb{C}[x]$ have a common root?

\[
\gcd\{f, g\} = 1 \iff \text{there exist } a, b \in \mathbb{C}[x] \text{ such that } af + bg = 1
\]

- theorem: can always choose $\deg a < \deg g$ and $\deg b < \deg f$
linear equations

suppose \( \deg f = l, \deg g = m \), and the above degree bounds
then the linear equation \( af + bg = 1 \) is

\[
\begin{bmatrix}
    f_0 & g_0 \\
    f_1 & g_1 \\
    \vdots & \vdots \\
    f_l & g_0 \\
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    \vdots \\
    a_{m-1} \\
    b_0 \\
    \vdots \\
    b_{l-1}
\end{bmatrix}
= 
\begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\]

this matrix is called the **Sylvester matrix** of \( f \) and \( g \), written \( \text{sy}l(f, g, x) \)
its determinant is called the **resultant** of \( f \) and \( g \), written \( \text{res}(f, g, x) \)
Suppose
\[ f = 2x^2 + 3x + 1 \quad g = 7x^2 + x + 3 \]
is \( 1 \in \text{ideal}\{f, g\} \), or equivalently, does \( \gcd\{f, g\} = 1 \)?

The resolvent is
\[
\text{res}(f, g, x) = \det \begin{bmatrix}
1 & 0 & 3 & 0 \\
3 & 1 & 1 & 3 \\
2 & 3 & 7 & 1 \\
0 & 2 & 0 & 7
\end{bmatrix} = 153
\]
since this is nonzero, we have \( \gcd\{f, g\} = 1 \).
multivariable polynomials

we can compute the resultant for multivariable polynomials, with respect to a single variable, e.g.,

\[ f = xy - 1 \quad g = x^2 + y^2 - 4 \]

to compute \( \text{res}(f, g, x) \), view \( f, g \) as polynomials in \( x \) with coeffs. in \( \mathbb{K}[y] \)

\[
\text{res}(f, g, x) = \det \begin{bmatrix}
-1 & 0 & -4 + y^2 \\
y & -1 & 0 \\
0 & y & 1
\end{bmatrix}
\]

\[ = y^4 - 4y^2 + 1 \]

\( \text{res}(f, g, x) \) eliminates \( x \) leaving a polynomial in \( y \)
example

with \( f = xy - 1 \) and \( g = x^2 + y^2 - 4 \) we have \( af + bg = 1 \) of appropriate degrees is equivalent to

\[
\begin{bmatrix}
-1 & 0 & -4 + y^2 \\
y & -1 & 0 \\
0 & y & 1
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
b_0
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

using the explicit formula for the matrix inverse gives

\[
\begin{bmatrix}
a_0 \\
a_1 \\
b_0
\end{bmatrix}
= \frac{1}{\text{res}(f, g, x)}
\begin{bmatrix}
-1 & -4y + y^3 & -4 + y^2 \\
-y & -1 & -4y + y^3 \\
y^2 & y & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

hence

\[
a = \frac{-xy - 1}{\text{res}(f, g, x)} \quad \quad b = \frac{y^2}{\text{res}(f, g, x)}
\]
so we have $f = xy - 1$ and $g = x^2 + y^2 - 4$ and

$$af + bg = 1$$

where

$$a = \frac{-xy - 1}{y^4 - 4y^2 + 1} \quad b = \frac{y^2}{y^4 - 4y^2 + 1}$$

multiplying by $\text{res}(f, g, x) = y^4 - 4y^2 + 1$ gives

$$\hat{a}f + \hat{b}g = \text{res}(f, g, x)$$

where $\hat{a} = -xy - 1$ and $\hat{b} = y^2$
elimination and resultants

we have

\[ \text{res}(f, g, x) \in \text{ideal}\{f, g\} \]

because the explicit formula for the matrix inverse gives

\[ \text{syl}(f, g, x_1)^{-1} = \frac{1}{\text{res}(f, g, x_1)} \text{adjoint}(\text{syl}(f, g, x_1))^T \]

and since the entries of \( \text{adjoint}(A) \) are polynomials in the entries of \( A \), the polynomials \( \hat{a} = a \text{res}(f, g, x) \) and \( \hat{b} = b \text{res}(f, g, x) \) satisfy

\[ \hat{a} f + \hat{b} g = \text{res}(f, g, x) \]
elimination and resultants

therefore the resultant is a member of the first elimination ideal

\[
f, g \in \mathbb{K}[x_1, \ldots, x_n] \implies \text{res}(f, g, x_1) \in I_1
\]

where \( I_1 = \text{ideal}\{f, g\} \cap \mathbb{K}[x_2, \ldots, x_n] \)

- implicitization of parameterized curves
- solution of two polynomial equations in two variables
another view of resultants

if $p(x_0) = q(x_0) = 0$ then

$$\begin{bmatrix}
p_n & p_{n-1} & \cdots & p_1 & p_0 \\
& p_n & & \cdots & \& \\
& & \ddots & & \ddots \\
q_m & q_{m-1} & \cdots & q_0 \\
& q_m & & \cdots & \&
\end{bmatrix} \begin{bmatrix}
x_{n+m-1} \\
x_{n+m-2} \\
\vdots \\
p(x_0)x_0 \\
p(x_0)x_0 \\
q(x_0)x_0^{n-1} \\
q(x_0)x_0^{n-2} \\
\vdots \\
q(x_0)x_0 \\
q(x_0)
\end{bmatrix} = 0$$
resultants and companion matrices

\[ \text{res}(p, q, x) = p_n^m \det q(C_p) \]

- no proofs today ...
for a univariate polynomial $p$, the discriminant is

$$\text{dis}(p) = (-1)^{\binom{n}{2}} \frac{1}{p_n} \text{res}(p, p', x)$$

- if $p$ and its derivative $p'$ have a common root, then $p$ has a root of multiplicity 2 or more
discriminants

- if $p = ax^2 + bx + c$ then $\text{dis}(p) = b^2 - 4ac$

- if $p = ax^3 + bx^2 + cx + d$ then

  $$\text{dis}(p) = -27a^2d^2 + 18adcb + b^2c^2 - 4b^3d - 4ac^3$$