Homework 1

1. **Axiomatic definition of a vector space.** A *vector space* is a set V together with a field F (we'll always use either $\mathbb{R}$ or $\mathbb{C}$), and two functions:

$$\begin{aligned} + : V \times V \to V \qquad & \text{vector addition} \\ \bullet : F \times V \to V \qquad & \text{scalar multiplication} \end{aligned}$$

which satisfy the following 8 axioms:

(i) *associativity:* for all $u, v \in V$

$$(u + v) + w = u + (v + w)$$

(ii) *identity:* there exists $\theta \in V$ such that for all $u \in V$

$$u + \theta = \theta + u = u$$

(iii) *inverse:* for all $u \in V$, there exists $x \in V$ such that

$$u + x = \theta$$

(iv) *commutativity:* for all $u, v \in V$,

$$u + v = v + u$$

(v) *associativity:* for all $\alpha, \beta \in F$ and $u \in V$

$$(\alpha\beta) \bullet u = \alpha \bullet (\beta \bullet u)$$

(vi) *identity:* for all $u \in V$

$$1 \bullet u = u$$

where 1 is the multiplicative identity element of F.

(vii) *distributivity:* for all $\alpha, \beta \in F$ and $u \in V$

$$(\alpha + \beta) \bullet u = \alpha \bullet u + \beta \bullet u$$

(viii) *more distributivity:* for all $\alpha \in F$ and $u, v \in V$

$$\alpha \bullet (u + v) = \alpha \bullet u + \alpha \bullet v$$

Notice that the requirement that $+$ maps $V \times V \to V$ implies that V is closed under addition, meaning that for all $u, v \in V$ we have $u + v \in V$. The first four axioms are called the *additive group axioms*.

We usually simplify notation in various ways. The vector θ of axiom (ii) is called the *zero vector*, and when we don't need to be careful about the distinction between $\theta \in V$ and $0 \in F$ we'll just write 0 instead of θ. Similarly we usually omit the symbol $\bullet$.

Using the axioms above, prove the following:

(a) The zero vector θ defined in (ii) is unique. Note that (iii) would be ambiguous otherwise. Hint: assume there exist θ_1, θ_2, that both satisfy (ii), and evaluate $\theta_1 + \theta_2$ in two different ways.

(b) If $u + v = u + w$, then $v = w$.

(c) Prove that $0 \bullet u = \theta$ for all $u \in V$.

(d) The additive inverse of a particular vector $u \in V$ is unique, and it is equal to $(-1) \bullet u$. Note that the notation $-u$ is defined to mean $(-1) \bullet u$.

2. **Subspaces.** A subset S of a vector space V is a *subspace* if it satisfies:

(i) *closure under vector addition:* $s_1 + s_2 \in S$ for all $s_1, s_2 \in S$.

(ii) *closure under scalar multiplication:* $\alpha s \in S$ for all $\alpha \in F$ and $s \in S$.

Suppose V is a vector space, and S, T are subspaces of V.

(a) Show that $S \cap T$ is a subspace.

(b) Define the sum of S and T to be

$$S + T = \{\, x + y \mid x \in S, y \in T \,\}$$

Show that $S + T$ is a subspace.

3. **Signals.**

(a) Define the signal $z : \mathbb{Z}_+ \to \mathbb{C}$ by

$$z_t = a^t \qquad \text{for all } t \geq 0$$

Prove that $z \in \ell_2$ if and only if $|a| < 1$.

(b) Define the signal $x : [0, \infty) \to \mathbb{C}$ by

$$x(t) = e^{at} \qquad \text{for all } t \geq 0$$

Prove that $x \in L_2$ if and only if $\mathrm{Re}(a) < 0$.

(c) For which values of $a \in \mathbb{R}$ is the function t^a an element of $L_2[0, 1]$? Give proofs or counterexamples as necessary.

4. **Induced norms.** Suppose U and V are normed spaces, and $A : U \to V$ is linear. The induced-norm of A is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

For each of the following parts, give a careful proof. In particular, make sure you distinguish between sup and max.

(a) Suppose A is a bounded linear map. Prove that

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$$

(b) Suppose $A : U \to V$ and $B : V \to W$ are bounded linear maps. Prove that

$$\|BAx\| \leq \|B\|\|A\|\|x\|$$

(c) Suppose $A \in \mathbb{R}^{n \times n}$ is a matrix defining a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ in the usual way. The Frobenius norm of A is

$$\|A\|_F = \big(\operatorname{trace}(A^T A)\big)^{\frac{1}{2}}$$

Show that the Frobenius norm is not induced by any choice of norm on $\mathbb{R}^n$. You may assume the same norm is used for both the domain and image spaces.

5. **The induced ∞-norm.** Show that the induced ∞-norm of a matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$\|A\|_{i\infty} = \max_i \sum_j |A_{ij}|$$

6. **Square-summable signals.**

(a) Show that, for any $x \in \ell_2(\mathbb{Z}_+, \mathbb{R})$ we have

$$\lim_{t \to \infty} x_t = 0$$

(b) Give a counterexample to show that the same is not true in $L_2([0, \infty), \mathbb{R})$.

7. **Norms.**

Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms.

(a) Define

$$\|x\|_c = \max\{\|x\|_a, \|x\|_b\}$$

Is $\|\cdot\|_c$ a norm?

(b) Answer the same question, replacing max by min above.

(c) Under what conditions on $A \in \mathbb{R}^{m \times n}$ is

$$\|x\|_A = \|Ax\|_\infty$$

a norm?