Homework 2

1. **Cauchy sequences.** Prove that every convergent sequence in a normed space is a Cauchy sequence.

2. **Completeness of $\ell_2$.** Suppose $x^0, x^1, x^2, \ldots$ is a sequence in $\ell_2$, that is each $x^i \in \ell_2$. Further suppose this sequence is Cauchy. We’d like to prove that it converges.

   (a) First fix an integer $k$, and consider the sequence of real numbers formed by the $k^{th}$ components of each vector. We’ll denote this sequence $x_0^k, x_1^k, x_2^k, \ldots$. Show that for each $k$ this sequence is Cauchy.

   (b) Using part (a) and the fact that $\mathbb{R}$ is complete, show that $\ell_2$ is complete.

3. **Open and closed sets.** For this question $S$ and $T$ are subsets of $V$, a normed space. Prove the following:

   (a) $S$ is closed if and only if the complement of $S$ is open.

   (b) If $S$ and $T$ are open, so is $S \cap T$ and $S \cup T$.

   (c) If $S$ and $T$ are closed, so is $S \cap T$ and $S \cup T$.

4. **Fourier Transforms.** Let $F$ be the fourier map $F : \ell_2(\mathbb{Z}) \to L_2(\mathbb{T})$. For each of the following functions $\hat{h} \in L_2(\mathbb{T})$, find the inverse $\lambda$-transform $h = F^* \hat{h}$.

   (a) $\frac{1}{1 - a\lambda}$ when $|a| < 1$

   (b) $\frac{1}{1 - a\lambda}$ when $|a| > 1$

   (c) $\lambda^2 + \lambda + 2$

   (d) $\lambda^{-1}$

   (e) The function

   $$u(e^{i\theta}) = \begin{cases} 
   1 & \text{if } |\theta| < \alpha \\
   0 & \text{otherwise}
   \end{cases}$$

   where $\alpha > 0$. 

---

1
Now suppose we wish to implement a linear system via convolution with \( h \), so that the output \( y \in \ell_2(\mathbb{Z}) \) is related to the input \( u \in \ell_2(\mathbb{Z}) \) by

\[
y_t = \sum_{k=-\infty}^{\infty} h_{t-k} u_k
\]

Which of these systems is causal?

5. **Causality**

For each \( t \in \mathbb{Z} \), let \( P_t \) be the projection operator defined by \( y = P_t u \) if

\[
y_k = \begin{cases} u_k & \text{if } k \leq t \\ 0 & \text{otherwise} \end{cases}
\]

Let \( G : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) \) be the convolution map defined by \( y = Gu \) if

\[
y_t = \sum_{k \in \mathbb{Z}} h_{t-k} u_k
\]

where \( h \in \ell_2(\mathbb{Z}) \). Show that \( G \) is lower triangular if

\[
P_t G = P_t GP_t \quad \text{for all } t \in \mathbb{Z}
\]

Give an interpretation of this equation.

6. **Least Squares in Hilbert space.**

We would like to approximate a function \( y \) using a linear combination of functions from a set \( \{a_1, \ldots, a_n\} \). Let’s formalize this idea.

Let \( E \) be the vector space \( \mathbb{R}^n \) equipped with the standard inner product:

\[
\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k
\]

and let \( H \) be a Hilbert space. Suppose \( a_1, \ldots, a_n \in H \). Define the linear operator \( A : E \to H \) as:

\[
Ax = \sum_{k=1}^{n} x_k a_k
\]

(a) Suppose \( x \in E \) is chosen in order to minimize \( \| y - Ax \|^2 \). Using calculus or otherwise, show that \( x \) must satisfy the normal equations:

\[
\begin{bmatrix}
\langle a_1, a_1 \rangle & \cdots & \langle a_1, a_n \rangle \\
\vdots & \ddots & \vdots \\
\langle a_n, a_1 \rangle & \cdots & \langle a_n, a_n \rangle
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
\langle a_1, y \rangle \\
\vdots \\
\langle a_n, y \rangle
\end{bmatrix}
\]

(1)
(b) Find the adjoint operator $A^*: H \to E$, and show that we can rewrite (1) as:

$$A^* Ax = A^* y$$

(c) We will now apply this method to approximate an impulse response. Choose $H$ to be the set $L_2$ with the standard inner product:

$$\langle f, g \rangle = \int_0^\infty f(t)g(t) \, dt$$

and consider the function $h \in L_2$ defined by:

$$h(t) = e^{-2t} \sin(4t)$$

Using Matlab, compute the linear combination of the functions:

$$a_k(t) = e^{-3t} t^{k-1} \quad \text{for } k = 1, 2, \ldots$$

that approximates $h$ as explained in part (a). For $n = 4$ and $n = 8$, plot $h$ and its approximation $h_{LS}$ on the same axes, and compute $\|h - h_{LS}\|$. *Hint:* you may want to look up the Matlab functions `sym`, `eval`, `int`, `ilaplace`.

(d) Let’s solve this problem in the frequency domain instead. Choose $H$ to be the set $H_2$ of stable transfer functions $f : j\mathbb{R} \to \mathbb{C}$, where $j\mathbb{R}$ means the imaginary axis in the complex plane. This space has the inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f(j\omega)}g(j\omega) \, d\omega$$

This time, consider the function $\hat{h} \in H_2$ defined by:

$$\hat{h}(s) = \frac{4}{s^2 + 4s + 20}$$

Note that this is the Laplace transform of the impulse response $h$ we used in part (c). Using Matlab, compute the linear combination of the functions:

$$\hat{a}_k(s) = \frac{1}{(s + 3)^k} \quad \text{for } k = 1, 2, \ldots$$

that approximates $\hat{h}$ as explained in part (a). For $n = 4$ and $n = 8$, plot $|\hat{h}(i\omega)|$ and its approximation $|\hat{h}_{LS}(i\omega)|$ on the same axes, and compute $\|\hat{h} - \hat{h}_{LS}\|$. You should find the same norm as in part (c).

(e) Verify that the approximations using both methods are in fact the same. That is, the Laplace transform of $h_{LS}$ is equal to $\hat{h}_{LS}$.