Homework 2

1. **Cauchy sequences.** Prove that every convergent sequence in a normed space is a Cauchy sequence.

2. **Completeness of ℓ_2.** Suppose $x^0, x^1, x^2, \ldots$ is a sequence in ℓ_2, that is each $x^i \in \ell_2$. Further suppose this sequence is *Cauchy*. We'd like to prove that it converges.

   (a) First fix an integer k, and consider the sequence of real numbers formed by the k^{th} components of each vector. We'll denote this sequence $x_k^0, x_k^1, x_k^2, \ldots$. Show that for each k this sequence is Cauchy.

   (b) Using part (a) and the fact that $\mathbb{R}$ is complete, show that ℓ_2 is complete.

3. **Open and closed sets.** For this question S and T are subsets of V, a normed space. Prove the following:

   (a) S is closed if and only if the complement of S is open.

   (b) If S and T are open, so is $S \cap T$ and $S \cup T$.

   (c) If S and T are closed, so is $S \cap T$ and $S \cup T$.

4. **Fourier Tramsforms.** Let F be the fourier map $F : \ell_2(\mathbb{Z}) \to L_2(\mathbb{T})$. For each of the following functions $\hat{h} \in L_2(\mathbb{T})$, find the inverse λ-transform $h = F^* \hat{h}$.

   (a) $\dfrac{1}{1 - a\lambda}$ when $|a| < 1$

   (b) $\dfrac{1}{1 - a\lambda}$ when $|a| > 1$

   (c) $\lambda^2 + \lambda + 2$

   (d) λ^{-1}

   (e) The function

   $$u(e^{j\theta}) = \begin{cases} 1 & \text{if } |\theta| < \alpha \\ 0 & \text{otherwise} \end{cases}$$

   where $\alpha > 0$.

Now suppose we wish to implement a linear system via convolution with h, so that the output $y \in \ell_2(\mathbb{Z})$ is related to the input $u \in \ell_2(\mathbb{Z})$ by

$$y_t = \sum_{k=-\infty}^{\infty} h_{t-k} u_k$$

Which of these systems is causal?

5. **Causality**

For each $t \in \mathbb{Z}$, let P_t be the projection operator defined by $y = P_t u$ if

$$y_k = \begin{cases} u_k & \text{if } k \leq t \\ 0 & \text{otherwise} \end{cases}$$

Let $G : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ be the convolution map defined by $y = Gu$ if

$$y_t = \sum_{k \in \mathbb{Z}} h_{t-k} u_k$$

where $h \in \ell_2(\mathbb{Z})$. Show that G is lower triangular if

$$P_t G = P_t G P_t \qquad \text{for all } t \in \mathbb{Z}$$

Give an interpretation of this equation.

6. **Least Squares in Hilbert space.**

We would like to approximate a function y using a linear combination of functions from a set $\{a_1, \ldots, a_n\}$. Let's formalize this idea.

Let E be the vector space $\mathbb{R}^n$ equipped with the standard inner product:

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$$

and let H be a Hilbert space. Suppose $a_1, \ldots, a_n \in H$. Define the linear operator $A : E \to H$ as:

$$Ax = \sum_{k=1}^{n} x_k a_k$$

(a) Suppose $x \in E$ is chosen in order to minimize $\|y - Ax\|^2$. Using calculus or otherwise, show that x must satisfy the *normal equations*:

$$\begin{bmatrix} \langle a_1, a_1 \rangle & \cdots & \langle a_1, a_n \rangle \\ \vdots & \ddots & \vdots \\ \langle a_n, a_1 \rangle & \cdots & \langle a_n, a_n \rangle \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \langle a_1, y \rangle \\ \vdots \\ \langle a_1, y \rangle \end{bmatrix} \tag{1}$$

(b) Find the adjoint operator $A^* : H \to E$, and show that we can rewrite (1) as:

$$A^*Ax = A^*y$$

(c) We will now apply this method to approximate an impulse response. Choose H to be the set L_2 with the standard inner product:

$$\langle f, g \rangle = \int_0^\infty f(t)g(t)\,\mathrm{d}t$$

and consider the function $h \in L_2$ defined by:

$$h(t) = e^{-2t}\sin(4t)$$

Using Matlab, compute the linear combination of the functions:

$$a_k(t) = e^{-3t}t^{k-1} \quad \text{for } k = 1, 2, \ldots$$

that approximates h as explained in part (a). For $n = 4$ and $n = 8$, plot h and its approximation h_{LS} on the same axes, and compute $\|h - h_{\mathrm{LS}}\|$. *Hint:* you may want to look up the Matlab functions `sym`, `eval`, `int`, `ilaplace`.

(d) Let's solve this problem in the frequency domain instead. Choose H to be the set H_2 of stable transfer functions $f : j\mathbb{R} \to \mathbb{C}$, where $j\mathbb{R}$ means the imaginary axis in the complex plane. This space has the inner product:

$$\langle f, g \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f(j\omega)}g(j\omega)\,\mathrm{d}\omega$$

This time, consider the function $\hat{h} \in H_2$ defined by:

$$\hat{h}(s) = \frac{4}{s^2 + 4s + 20}$$

Note that this is the Laplace transform of the impulse response h we used in part (c). Using Matlab, compute the linear combination of the functions:

$$\hat{a}_k(s) = \frac{1}{(s+3)^k} \quad \text{for } k = 1, 2, \ldots$$

that approximates $\hat{h}$ as explained in part (a). For $n = 4$ and $n = 8$, plot $|\hat{h}(i\omega)|$ and its approximation $|\hat{h}_{\mathrm{LS}}(i\omega)|$ on the same axes, and compute $\|\hat{h} - \hat{h}_{\mathrm{LS}}\|$. You should find the same norm as in part (c).

(e) Verify that the approximations using both methods are in fact the same. That is, the Laplace transform of h_{LS} is equal to $\hat{h}_{\mathrm{LS}}$.