Homework 3

1. **Null space and range of an operator.** Let $A : U \to V$ be a bounded linear operator where U and V are Hilbert spaces. The *null space* and *range* of A are defined as

$$\begin{aligned} \operatorname{null} A &= \{\, x \in U \mid Ax = 0 \,\} \\ \operatorname{range} A &= \{\, Ax \mid x \in U \,\}. \end{aligned}$$

   (a) Show that $\operatorname{null} A$ is a subspace of U and $\operatorname{range} A$ is a subspace of V.

   (b) Show that $\operatorname{null} A$ is a closed subspace.

   (c) In general, $\operatorname{range} A$ is not closed. Give an example of an operator with closed range, and another with range that is not closed.

   (d) The map A is called ***bounded below*** if there exists $m > 0$ such that for all x

$$\|Ax\| \geq m\|x\|$$

   Show that if A is bounded below then $\operatorname{range} A$ is closed.

   (e) Suppose $\hat{g} \in L_2(\mathbb{T})$ is continuous, and $g(\lambda) \neq 0$ for all $\lambda \in \mathbb{T}$. Is $\operatorname{range} A$ closed? Give a proof or counterexample.

   (f) What is the range of the multiplication operator $M_{\hat{g}} : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ when $\hat{g}(\lambda) = \lambda^k$?

2. **The induced norm of a semi-infinite Toeplitz map.** Suppose $\hat{g} \in L_\infty(\mathbb{T}) \cap H_2(\mathbb{T})$, with Fourier coefficients $g_0, g_1, g_2, \ldots$. Define the Toeplitz map

$$S_g = \begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ g_2 & g_1 & g_0 & \\ \vdots & & & \ddots \end{bmatrix}$$

   (a) Show that the induced ℓ_2-norm of S_g is

$$\|S_g\| = \|\hat{g}\|_\infty$$

   (b) Suppose $\hat{g}(\lambda) = a\lambda + b\lambda^2$ where $a, b \in \mathbb{R}$. Find $\|S_g\|$.

3. **Multiplication operators.** Suppose $\hat{g} \in L_2(\mathbb{T})$, and let $M_{\hat{g}} : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be the associated multiplication operator.

(a) Under what conditions is $M_{\hat{g}}$ unitary?

(b) Under what conditions is $M_{\hat{g}}$ Hermitian?

4. **State-space represenatations.** Suppose $\hat{g} \in L_2(\mathbb{T})$ is given by

$$\hat{g}(\lambda) = C(\lambda^{-1}I - A)^{-1}B + D$$

What is the inverse Fourier transform $\ldots, g_{-2}, g_{-1}, g_0, g_1, g_2, \ldots$ when

(a) $\rho(A) < 1$

(b) all eigenvalues λ of A satisfy $|\lambda| > 1$

(c) all eigenvalues λ of A satisfy $|\lambda| \neq 1$

5. **The range of a Toeplitz map.** Suppose $\hat{g} \in L_2(\mathbb{T})$, and let $S_g : \ell_2(\mathbb{Z}_+) \to \ell_2(\mathbb{Z}_+)$ be the associated Toeplitz operator

$$S_g = \begin{bmatrix} g_0 & g_{-1} & & \\ g_1 & g_0 & g_{-1} & \\ g_2 & g_1 & g_0 & \\ \vdots & & & \ddots \end{bmatrix}$$

For each of the following cases, is it true that range $S_g = \ell_2(\mathbb{Z}_+)$? Give a proof or counterexample in each case.

(a) $1 - \dfrac{1}{2}\lambda$

(b) $1 - 2\lambda$

(c) $\dfrac{1}{1 - \frac{1}{2}\lambda}$

(d) $\dfrac{\lambda - \frac{1}{2}}{1 - \frac{1}{2}\lambda}$