Homework 5

Due: Dec. 9

The goal of this assignment is to develop code to solve a steady-state Kalman filtering problem. First we will implement the polynomial method described in class, and then we will implement the standard state-space method.

1. **Polynomials.** For the following parts, you may use Maple, Mathematica, or Matlab to produce the required code.

   (a) Spectral factorization: write a function that takes as input a rational function $\hat{g}$, and returns a rational function $\hat{p}$ such that $\hat{g}(\lambda)\overline{\hat{g}(\lambda)} = \hat{p}(\lambda)\overline{\hat{p}(\lambda)}$ for all $\lambda \in \mathbb{T}$, and all the roots and zeros of $\hat{p}$ lie *outside* the unit disc. Provide a couple test cases to show your function works.

   (b) Positive projection: write a function that takes as input a rational function $\hat{g}$, and decomposes it as: $\hat{g} = \hat{g}_+ + \hat{g}_-$, where $\hat{g}_+ \in H_2$ and $\hat{g}_- \in H_2^\perp$. Provide a couple test cases to show your functions works.

   (c) Least squares: recall from class that given $g, h \in \ell_2(\mathbb{Z}_+)$, the solution to the least-squares problem

$$\begin{aligned} \text{minimize:} \quad & \left\| h - S_g k \right\|_2 \\ \text{subject to:} \quad & k \in \ell_2(\mathbb{Z}_+) \end{aligned} \tag{1}$$

   is given by:

$$k = \begin{bmatrix} 0 & I \end{bmatrix} L_p^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} L_{\tilde{p}}^{-1} \begin{bmatrix} 0 \\ S_g h \end{bmatrix}$$

   where p is derived from g as in part (a). Using your functions from parts (a) and (b), write a function that takes as input the rational functions $\hat{g}$ and $\hat{h}$, and returns the rational function $\hat{k}$.

   (d) Consider the following transfer functions on $L_2(\mathbb{T})$:

$$\hat{g}(\lambda) = \frac{(2\lambda + 1)(\lambda^2 + 2\lambda + 5)}{(\lambda - 3)(\lambda^2 - 6\lambda + 25)} \qquad \hat{h}(\lambda) = \frac{-\lambda(3\lambda - 1)(4\lambda - 1)}{(\lambda - 3)(\lambda^2 - 6\lambda + 25)}$$

   Use your result from part (c) to find the optimal $\hat{k}$. Express your answer as a rational function, and also compute the first 20 coefficients of the filter's impulse response: $k_0, k_1, \cdots, k_{19}$.

2. **State-space.** For the following parts, use Matlab when asked to produce code.

   (a) Consider the standard state-space form for a stable SISO system:

   $$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned}$$

   where $\rho(A) < 1$. Compute the impulse response vector g, and show that its Fourier transform is equal to:

   $$\hat{g} = C\big(\lambda^{-1}I - A\big)^{-1}B + D$$

   (b) Consider the following state-space system:

   $$\begin{aligned} x(t+1) &= Ax(t) + Gw(t) \\ y_1(t) &= C_1x(t) \\ y_2(t) &= C_2x(t) + Hw(t) \end{aligned}$$

   where $\rho(A) < 1$ and the signals y_1, y_2, and w are scalars. Note that w is an unknown noise disturbance (not an input). Suppose we get to measure the output y_2, and we would like to estimate the signal y_1. In other words, we want to find an estimator of the form:

   $$\begin{aligned} z(t+1) &= A_Kz(t) + B_Ky_2(t) \\ \tilde{y}_1(t) &= C_Kz(t) + D_Ky_2(t) \end{aligned}$$

   that minimizes the expected value $\lim_{t\to\infty} \mathrm{E}\|y_1(t) - \tilde{y}_1(t)\|_2^2$ where w is Gaussian white noise. Using part (a), show how to turn this state-space problem into a least-squares problem of the form (1). In other words, show how to construct g, h from A, C_1, C_2, G, H, and show how k relates to A_K, B_K, C_K, D_K.

   (c) The problem from part (b) is a Kalman filtering problem. Write code that takes in matrices A, C_1, C_2, G, H, and produces the optimal filter A_K, B_K, C_K, D_K. You may use the Matlab function `kalman`.

   (d) Find a state-space representation for problem 1(d), and use your work from 2(c) to solve this problem in state-space. Using 2(a), convert this state space representation for the controller into a transfer function $\hat{k}$. Verify that it matches the result from problem 1(d).

   (e) Choose a finite horizon $N = 50$, and simulate the system driven by noise. Let:

   ```
   N = 50;
   w = randn(N,1);
   ```

   on the same axes, plot the output of the system driven by noise: S_hw, and also plot the output of the optimal estimator S_kS_gw.

3. **Minimal Realizations.** Find minimal state-space realizations for

   $$G(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ \frac{1}{s^2} & \frac{1}{s} \end{bmatrix} \qquad H(s) = \begin{bmatrix} 0 & \frac{1}{s} \\ \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$$