1 Norms and Vector Spaces

Suppose we have a complex vector space V. A ***norm*** is a function $f : V \to \mathbb{R}$ which satisfies

(i) $f(x) \geq 0$ for all $x \in V$

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in V$

(iii) $f(\lambda x) = |\lambda| f(x)$ for all $\lambda \in \mathbb{C}$ and $x \in V$

(iv) $f(x) = 0$ if and only if $x = 0$

Property (ii) is called the *triangle inequality*, and property (iii) is called *positive homgeneity*. We usually write a norm by $\|x\|$, often with a subscript to indicate which norm we are refering to. For vectors $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$ the most important norms are as follows.

- The ***2-norm*** is the usual Euclidean length, or RMS value.

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

- The ***1-norm***

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

- For any integer $p \geq 1$ we have the p***-norm***

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

- The ∞***-norm***, also called the *sup-norm*. It gives the peak value.

$$\|x\|_\infty = \max_i |x_i|$$

  This notation is used because $\|x\|_\infty = \lim_{p \to \infty} \|x\|_p$.

One can show that these functions each satisfy the properties of a norm. The norms are also nested, so that

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$$

This is easy to see; just sketch the unit ball

$$\{ \, x \in \mathbb{R}^2 \mid \|x\| \leq 1 \, \}$$

for each of the norms, and notice that they are nested.

These norms also satisfy pairwise inequalities; for example

$$\|x\|_1 \leq n \|x\|_\infty \qquad \text{for all } x \in \mathbb{C}^n$$

In fact, in finite-dimensional vector spaces such inequalities hold between any pair of norms. So if one designs a controller or an estimator to make a particular norm small, then one is simultaneously squeezing all the other norms also (but not necessarily optimally).

1.1 Infinite-dimensional vector spaces

Vector spaces are defined by the usual axioms of addition and scalar multiplication. The important spaces are as follows. Note that there are real-valued versions of all of these spaces.

Sequence space. Define the space

$$\ell_e = \{\, x : \mathbb{Z}_+ \to \mathbb{C} \,\}$$

This is an infinite-dimensional vector space. (The subscript e stands for *extended*, and we'll see why that's used later in the course.) We think about this vector space as the space of sequences, or of *signals* in discrete-time.

The square-summable sequence space ℓ_2. We need a norm to make ℓ_e useful. For some vectors $x \in \ell_e$ we can define

$$\|x\|_2 = \left(\sum_{i=0}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$$

but there are of course vectors $x \in \ell_e$ for which the series doesn't converge. Define ℓ_2 to be those x for which it does converge

$$\ell_2 = \{\, x \in \ell_e \mid \|x\| \text{ is finite} \,\}$$

For example, the signal $x(k) = a^k$ is an element of ℓ_2 if and only if $|a| < 1$. The perhaps surprising fact is that ℓ_2 is a *subspace* of ℓ_e. Recall that a set S is a subspace if and only if

(i) $x + y \in S$ for all $x, y \in S$

(ii) $\lambda x \in S$ for all $x \in S$ and $\lambda \in \mathbb{C}$

that is, a subspace is a set which is closed under addition and scalar multiplication. Closure under scalar multiplication is easy; let's prove closure under addition.

Theorem 1. *Suppose $x, y \in \ell_2$. Then $x + y \in \ell_2$ and*

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. We have

$$\begin{aligned} \left(\sum_{i=0}^{n} |x_i + y_i|^2 \right)^{\frac{1}{2}} &\leq \left(\sum_{i=0}^{n} |x_i|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=0}^{n} |y_i|^2 \right)^{\frac{1}{2}} \\ &\leq \|x\| + \|y\| \end{aligned}$$

where the first inequality follows from the triangle inequality for vectors in $\mathbb{C}^{n+1}$. We therefore have that the partial sum

$$s_n = \sum_{i=0}^{n} |x_i + y_i|^2$$

is bounded as a function of n, and since it is non-decreasing and bounded it must converge. Therefore the series

$$\sum_{i=0}^{\infty}|x_i + y_i|^2$$

converges, and $x + y \in \ell_2$. The triangle inequality also follows. ■

Variants of ℓ_2. We'll have need for many variants of ℓ_2, such as

- *bi-infinite sequences*

$$\ell_2(\mathbb{Z}) = \left\{ x : \mathbb{Z} \to \mathbb{C} \;\middle|\; \sum_{i=-\infty}^{\infty} |x_i|^2 \text{ is finite} \right\}$$

- *vector-valued sequences*

$$\ell_2(\mathbb{Z}_+, \mathbb{C}^n) = \left\{ x : \mathbb{Z}_+ \to \mathbb{C}^n \;\middle|\; \sum_{i=0}^{\infty} \|x_i\|_2^2 \text{ is finite} \right\}$$

- *general sequences.* Let $D \subset \mathbb{Z}^m$ and

$$\ell_2(D, \mathbb{C}^n) = \left\{ x : D \to \mathbb{C}^n \;\middle|\; \sum_{i \in D} \|x_i\|_2^2 \text{ is finite} \right\}$$

ℓ_p spaces. The general ℓ_p spaces are defined similarly, with the p-norm replacing the 2-norm. In particular, for $x : \mathbb{Z}_+ \to \mathbb{C}$ the ∞-norm is defined as

$$\|x\|_\infty = \sup_{t \in \mathbb{Z}_+} |x(t)|$$

The ℓ_p spaces are nested; that is

$$\ell_1 \subset \ell_2 \subset \ell_\infty$$

The L_2 function spaces. Define the vector space

$$L_2([0,1]) = \{\, x : [0,1] \to \mathbb{C} \mid x \text{ is Lebesgue measurable and } \|x\|_2 \text{ is finite} \,\}$$

where the norm is

$$\|x\|_2 = \left(\int_0^1 |x(t)|^2 \, dt \right)^{\frac{1}{2}}$$

The technical requirement of Lebesgue measurability will not be a concern for us. The most common L_2 space fwor us will be

$$L_2([0,\infty)) = \{\, x : [0,\infty) \to \mathbb{C} \mid x \text{ is Lebesgue measurable and } \|x\|_2 \text{ is finite} \,\}$$

For example, $x(t) = e^{at}$ is an element of $L_2([0,\infty))$ if and only if $\mathrm{Re}(a) < 0$. More generally, suppose $D \subset \mathbb{C}^m$ and define

$$L_2(D, \mathbb{C}^m) = \{\, f : D \to \mathbb{C}^m \mid \|f\|_2 \text{ is finite} \,\}$$

where the norm is

$$\|f\|_2 = \left(\int_{t \in D} \|f(t)\|_2^2 \, dt \right)^{\frac{1}{2}}$$

The most common cases are $D = [0,1]$, $D = [0,\infty)$ and $D = (-\infty,\infty)$. Again, one can prove that L_2 is a vector space; that is, it is closed under addition and scalar multiplication.

The L_p function spaces. These are defined similarly, with

$$\|x\|_p = \left(\int_0^1 |x(t)|^p \, dt \right)^{\frac{1}{p}}$$

for $p \geq 1$ and

$$\|x\|_\infty = \operatorname*{ess\,sup}_{t \in D} |f(t)|$$

Here ess sup means *essential supremum*; it is the sup of f over all but a set of measure zero. Again, the measure theory won't matter to us. As before, for functions of time we think about the 2-norm as the RMS value of the signal and the ∞-norm as its peak. We have the nesting

$$L_\infty([0,1]) \subset L_2([0,1]) \subset L_1([0,1])$$

Note that this nesting doesn't hold for $L_p(\mathbb{R})$. There is no constant K such that for all $x \in L_2([0,\infty)) \cap L_\infty([0,\infty))$

$$\|x\|_2 \leq K \|x\|_\infty$$

nor is there any constant K such that

$$\|x\|_\infty \leq K \|x\|_2$$

Unlike finite-dimensional spaces, such inequalities do not hold between any pair of norms. So minimizing the 2-norm is very different from minimizing the ∞-norm.

Functions on the complex plane. An important space in control theory is RL_2, the space of ***rational functions*** with no poles on the complex unit circle. This is a vector space, and we use the norm

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{j\theta}) \right|^2 d\theta \right)^{\frac{1}{2}}$$

Similarly, the space RH_2 is the space of rational functions with no poles in $\bar{\mathbb{D}}$, where

$$\bar{\mathbb{D}} = \{\, z \in \mathbb{C} \mid |z| \leq 1 \,\}$$

Again, this is a vector space, with the same norm as RL_2.

1.2 Properties of the norm

Suppose V is a normed space; that is a vector space equipped with a norm.

Lemma 2. *For any $x, y \in V$ we have*

$$\|x\| - \|y\| \leq \|x - y\|$$

Proof. This is a consequence of the triangle inequality. We have

$$\|x\| - \|y\| = \|x - y + y\| - \|y\| \leq \|x - y\| + \|y\| - \|y\| = \|x - y\|$$

■

Lemma 3. *The norm is continuous.*

Proof. At any point $a \in V$, we have

$$\big| \|a + x\| - \|a\| \big| \leq \|x\|$$

from Lemma 2. Hence we can make the norm of $a + x$ as close as we need to $\|a\|$ by making $\|x\|$ small. Hence the norm is continuous at a, and this is true for all $a \in V$. ■

Another important property is that every norm is a convex function, and has convex sublevel sets.

1.3 Linear maps

Suppose U and V are normed spaces; Consider the set of all possible linear maps

$$F_{\text{linear}}(U, V) = \{\, f : U \to V \mid f \text{ is linear} \,\}$$

This is a vector space. We define the *induced norm* of a linear map $A : U \to V$ by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Note that the norm of Ax is the norm in the space V, and the norm of x is the norm in the space U, and these norms may be different. We then define

$$L(U, V) = \{\, A \in F_{\text{linear}}(U, V) \mid \|A\| \text{ is finite} \,\}$$

If $\|A\|$ is finite, then A is called a ***bounded*** linear map, otherwise A is called ***unbounded***. The space $L(U, V)$ is called the space of bounded linear maps from U to V. It is easy to see that the norm is also given by

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$$

If U and V are finite dimensional, then every linear map $A : U \to V$ is bounded, because in finite dimensional spaces the unit ball is compact. Also the map $x \mapsto Ax$ is continuous, since we can write it in a basis as matrix multiplication, and the norm is continuous, so the composition $x \mapsto Ax$ is also continuous. Hence the induced norm of A is the maximum of a continuous function over a compact set, and so the maximum is attained.

The induced 2-norm. Suppose $A \in \mathbb{R}^{m \times n}$ is a matrix, which defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ in the usual way. Then the induced 2-norm of A is

$$\|A\| = \sigma_1(A)$$

where σ_1 is the largest singular value of the matrix A. This is also called the ***spectral norm*** of A, and occasionally written as

$$\|A\|_{i2}$$

where $i2$ stands for *induced 2-norm*.

The induced ∞-norm. Suppose $A \in \mathbb{R}^{m \times n}$. The induced ∞-norm of A is

$$\|A\|_{i\infty} = \max_i \sum_j |A_{ij}|$$