This section is from Stephen Boyd's notes for the course *Multivariable Control*. Alternative references are Callier and Desoer, or Kailath.

8 Smith-McMillan Form

8.1 Standard Notation

The set $\mathbb{R}[z]$ is the set of ***polynomials*** with coefficients in $\mathbb{R}$, and the set $\mathbb{R}(z)$ is the set of ***rational functions***. The 'z' in this notation is a convention from algebra. We also use $\mathbb{R}[z]^{m\times n}$ to denote the set of $m \times n$ matrices whose entries are polynomials.

8.2 Normal Rank of a Rational Matrix

If $A \in \mathbb{R}[z]^{m\times n}$ then the rank depends on z. For example

$$\operatorname{rank}\begin{bmatrix} \frac{z-1}{z+1} & 0 \\ 0 & \frac{(z-1)(z+2)}{(z+1)^2} \end{bmatrix} = \begin{cases} 0 & \text{if } z = 1 \\ 1 & \text{if } z = -2 \\ \text{meaningless} & \text{if } z = -1 \\ 2 & \text{otherwise} \end{cases}$$

We define the ***normal rank*** of A to be the maximum rank of $A(z)$ over all $z \in \mathbb{C}$. For example, if $A \in \mathbb{R}^{n\times n}$, then

$$\text{normal rank}(zI - A) = n$$

For all but finitely many $z \in \mathbb{C}$

$$\operatorname{rank} A(z) = \text{normal rank}\, A(z)$$

since the determinant of any minor of A, which is a rational function, must either vanish identically or vanish for only finitely many $z \in \mathbb{C}$.

A square polynomial matrix $U \in \mathbb{R}[z]^{n\times n}$ is called ***unimodular*** if

$$\det U(z) \neq 0 \text{ for all } z \in \mathbb{C}$$

Theorem 1. *Suppose $U \in \mathbb{R}[z]^{n\times n}$. Then U is unimodular if and only if there exists a nonzero constant $c \in \mathbb{R}$ such that*

$$\det U = c$$

Proof. The *if* direction is clear. The *only if* holds because $\det U$ is a polynomial, which by assumption is nonzero for all $z \in \mathbb{C}$, and hence must be constant. ■

Notice the notation is meant to indicate that $\det U$ is a polynomial. When we write $\det U(z)$, it means that determinant is evaluated first, *before* evaluating U at a particular z. This means that, for example,

$$\det \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & z \end{bmatrix} = 1$$

and is not undefined when $z = 0$.

Theorem 2. *Suppose $U \in \mathbb{R}[z]^{n\times n}$. Then U is unimodular if and only if*

$$U^{-1} \in \mathbb{R}[z]^{n\times n}$$

Proof. First, we show *only if*. Suppose U is unimodular. Then $\det U$ is a nonzero constant, and so by Cramer's rule

$$U^{-1} = \frac{1}{\det U} \operatorname{adj} U$$

and $\operatorname{adj} U$ is a polynomial matrix. Conversely, suppose $U^{-1} \in \mathbb{R}[z]^{n\times n}$. Then

$$\det U = \frac{1}{\det U^{-1}}$$

Since both $\det U$ and $\det U^{-1}$ are polynomials, they are constant. ∎

The two most important unimodular matrices are

$$U_1 = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

which simply scales a row by c, and

$$U_1(z) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & q(z) & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

where exactly one entry is a polynomial, and the diagonal entries are all 1. This matrix adds q times the j'th row to the i'th row. These two matrices are called ***elementary*** matrices.

Multiplication by unimodular matrices does not change the rank at all, so that

$$\operatorname{rank} UAV = \operatorname{rank} A \qquad \text{for all } z \in \mathbb{C}$$

Note that this is *rank*; it's a much stronger property than simply preservation of *normal rank*.

8.3 Smith Form

Suppose $M \in \mathbb{R}[z]^{p\times q}$, and normal rank $M = r$. There exist square unimodular matrices L and R such that

$$LMR = \left[\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ \hline & & & 0 \end{array}\right]$$

where $\lambda_i \in \mathbb{R}[z]$ are monic, and λ_i divides into λ_{i+1}. Here L is $p\times p$ and R is $q \times q$, so that zero block in the bottom right corner of the matrix LMR is $(p-r)\times(q-r)$.

The λ_i are called the ***invariant polynomials*** of M. They are uniquely defined by M, in particular

$$\lambda_i = \frac{\Delta_i}{\Delta_{i-1}}$$

where $\Delta_0 = 1$ and

$$\Delta_i = \text{monic GCD of all } i \times i \text{ minors}$$

8.4 Smith-MacMillan Form

Suppose now $H \in \mathbb{R}(z)^{p\times q}$ is a proper rational matrix. Let

$$d = \text{monic LCM of denominators of all entries of } H$$

and define the polynomial matrix N by

$$H = \frac{N}{d}$$

Since N is a polynomial matrix, let it's Smith form be

$$N = U_1 S U_2$$

where U_i are square and unimodular, and

$$S = \left[\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ \hline & & & 0 \end{array}\right]$$

Then

$$H = U_1 \frac{S}{d} U_2$$

and we can cancel common factors from the numerators and denominators of the entries in S/d to give

$$\frac{S}{d} = \left[\begin{array}{ccc|c} \frac{\varepsilon_1}{\psi_1} & & & \\ & \ddots & & \\ & & \frac{\varepsilon_r}{\psi_r} & \\ \hline & & & 0 \end{array}\right]$$

Here

$$\frac{\varepsilon_i}{\psi_i} = \frac{\lambda_i}{d}$$

The polynomials ε_i and ψ_i are coprime, so we have

$$\varepsilon_i \text{ divides into } \varepsilon_{i+1} \quad \text{and} \qquad \psi_{i+1} \text{ divides into } \psi_i$$

The decomposition

$$H = U_1 \left[\begin{array}{ccc|c} \frac{\varepsilon_1}{\psi_1} & & & \\ & \ddots & & \\ & & \frac{\varepsilon_r}{\psi_r} & \\ \hline & & & 0 \end{array}\right] U_2$$

is called the ***Smith-MacMillan*** form of H.

For example, if

$$H(z) = \begin{bmatrix} \frac{z-1}{z+1} & \frac{z-1}{(z+1)^2} \\ 0 & \frac{z+2}{z^2-1} \end{bmatrix}$$

then

$$d(z) = (z-1)(z+1)^2 \qquad N(z) = \begin{bmatrix} (z-1)^2(z+1) & (z-1)^2 \\ 0 & (z+1)(z+2) \end{bmatrix}$$

Now $\Delta_1 = 1$ and $\Delta_2 = \det N = (z-1)^2(z+1)^2(z+2)$, and so the Smith form of N is

$$\begin{bmatrix} 1 & 0 \\ 0 & (z-1)^2(z+1)^2(z+2) \end{bmatrix}$$

and hence the Smith-MacMillan form of H is

$$\begin{bmatrix} \frac{1}{(z-1)(z+1)^2} & 0 \\ 0 & \frac{(z-1)(z+2)}{1} \end{bmatrix}$$

8.5 Poles and Zeros

The ***poles*** of H are defined to be the zeros of the polynomial

$$\prod_{i=1}^{r} \psi_i$$

and this gives their multiplicities also. If we don't care about multiplicities, then the poles are the zeros of ψ_1.

The ***zeros*** of H are defined to be the zeros of the polynomial

$$\prod_{i=1}^{r} \varepsilon_i$$

and again if we don't care about multiplicities, then the poles are the zeros of ε_r.

For the example above, H has zeros $(1, -2)$ and poles $(1, -1, -1)$. In particular, it has both a zero and a pole at 1.

The poles of H are just the poles of the entries of H since

$$\psi_1 = d$$

and d is the LCM of the entries of H. But the zeros are not so easy to characterize in terms of the entries of H. For the example above

$$\det H = \frac{z+2}{(z+1)^2}$$

and so we can have $\det H(\lambda) \neq 0$ even when λ is a zero of H. Also N drops rank at $z = -1$, which is not a zero of H.

8.6 Directions associated with poles and zeros

Suppose H has Smith-MacMillan form

$$H = U_1 \left[\begin{array}{ccc|c} \frac{\varepsilon_1}{\psi_1} & & & \\ & \ddots & & \\ & & \frac{\varepsilon_r}{\psi_r} & \\ \hline & & & 0 \end{array}\right] U_2$$

and λ is zero of H, say

$$\varepsilon_k(\lambda) = 0 \text{ but } \varepsilon_{k+1}(\lambda) = 0$$

Then $\psi_{k+1}(\lambda) \neq 0$, but it is possible that $\psi_k(\lambda) = 0$. Partition $U_1(\lambda)$ and $U_2(\lambda)$ as follows.

$$U_1(\lambda) = \begin{bmatrix} \hat{U}_1(\lambda) & U_{\text{out}}(\lambda) \end{bmatrix} \qquad U_2 = \begin{bmatrix} \hat{U}_2(\lambda) \\ U_{\text{in}}^*(\lambda) \end{bmatrix}$$

Note that these are evaluated at λ, and so are simply complex matrices. Then we define

$$\begin{aligned} \text{range}\, U_{\text{in}}(\lambda) &= \text{ input zero space} \\ \text{range}\, U_{\text{out}}(\lambda) &= \text{ output zero space} \end{aligned}$$

These vector spaces do not depend on the particular unimodular matrices used to reduce H to Smith-MacMillan form.