EE365: Costs and Rewards

Costs and rewards

Value iteration

Costs and rewards

Costs and rewards in a Markov chain

- associate costs (or rewards; more generally, just a function) with Markov chain $x_0, \ldots, x_T$
- $g_t : \mathcal{X} \to \mathbb{R}$ is the stage cost function
- at time t, we incur cost $g_t(x)$ for being in state x
- total cost for T time periods is (random variable) $\sum_{t=0}^{T} g_t(x_t)$
- expected stage cost is $\pi_t g_t$
- expected total cost is (number)

$$J = \mathbf{E} \sum_{t=0}^{T} g_t(x_t) = \pi_0 g_0 + \cdots + \pi_T g_T$$

Cost evaluation by distribution propagation

- $J = \pi_0 g_0 + \cdots + \pi_T g_T$
- evaluate π_t recursively using distribution propagation
- start with $J = \pi_0 g_0$, then for $t = 1, \ldots, T$,

$$\pi_t = \pi_{t-1} P \qquad \text{// propagate distribution forward in time}$$
$$J = J + \pi_t g_t \qquad \text{// running sum of expected stage costs}$$

- requires $n^2 T$ operations (less if P is sparse)

Value iteration

Value function

write J as

$$\begin{aligned}
J &= \pi_0 g_0 + \cdots + \pi_T g_T \\
&= \pi_0 g_0 + \cdots + \pi_0 P^T g_T \\
&= \pi_0 (\underbrace{g_0 + P g_1 + \cdots + P^T g_T}_{v_0}) \\
&= \pi_0 (g_0 + P(\underbrace{g_1 + P g_2 + \cdots + P^{T-1} g_T}_{v_1})) \\
&\vdots \\
&= \pi_0 (g_0 + P(g_1 + \cdots + P(g_{T-1} + P \underbrace{g_T}_{v_T})))
\end{aligned}$$

Value function

- define
$$v_t = g_t + Pg_{t+1} + \cdots + P^{T-t}g_T, \quad t = 0, \ldots, T$$
- $v_t : \mathcal{X} \to \mathbb{R}$ is *value function* at time t
- $J = \pi_0 v_0$; more generally,
$$J = \sum_{t=0}^{s-1} \pi_t g_t + \pi_s v_s$$
- first term is expected cost over $t = 0, \ldots, s-1$
- second term is expected cost over $t = s, \ldots, T$

Interpretation of value function

- we have

$$(v_t)_i = \mathbf{E}\left(\sum_{\tau=t}^{T} g_\tau(x_\tau) \;\middle|\; x_t = i \right)$$

- so v_t gives expected future cost starting from each state, at time t
- v_t summarizes future costs as a current cost

Recursion for value function

- from the definition of v_t we have $v_T = g_T$ and

$$v_{t-1} = g_{t-1} + P v_t, \quad t = T, \ldots, 1$$

- gives a *backward* recursion for computing $v_T, \ldots, v_0$
- called *value iteration*

Cost evaluation by value iteration

- start with $v_T = g_T$, then for $t = T, \ldots, 1$,

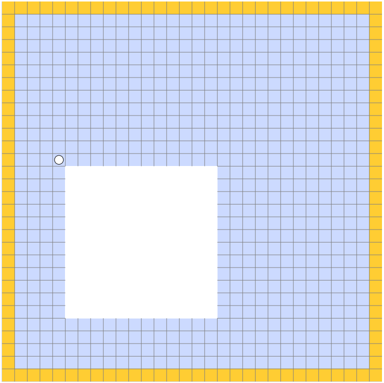
$$v_{t-1} = g_{t-1} + Pv_t \qquad // \text{ propagate value function backward in time}$$

- let $J = \pi_0 v_0$

- requires $n^2 T$ operations (less if P is sparse)
- an alternative to distribution propagation, that we will need for control

Example: Random walk

- random walk on a 2-dimensional 30×30 grid, with square obstacle
- outer boundaries are absorbing



Transition probabilities

2 different cases:

probability of staying at current state: $1/10$

Example: Mean time to absorption

- Let E be the set of absorbing states, and

$$\tau = \min\{t > 0 \mid x_t \in E\}$$

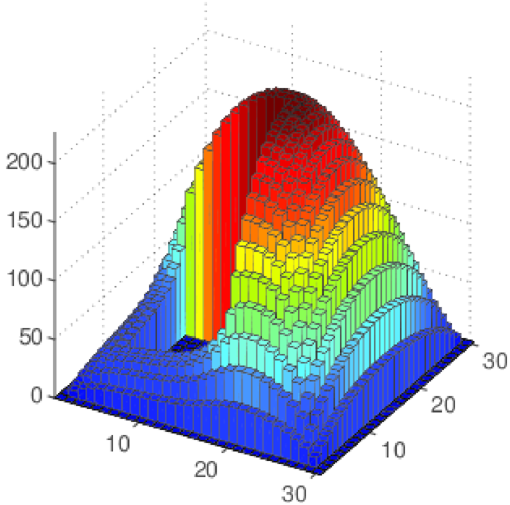
- for $t = 0, \ldots, T$ assign costs

$$g_t(x) = \begin{cases} 0 & x \in E \\ 1 & \text{otherwise} \end{cases}$$

- cost $J = \mathbf{E} \min(\tau, T)$
- gives mean time to absorption as $T \to \infty$

Example: Mean time to absorption

mean time to absorption as a function of initial state



Example: Mean time in each state

- pick state j, let

$$g_t(x) = \begin{cases} 1 & x = j \\ 0 & \text{otherwise} \end{cases}$$

- then J is the mean time spent in state j during time $t \in [0, T]$

Example: Mean time in each state

plot shows the mean time spent in non-absorbing states (initial state $i = (12, 18)$)