EE365: Linear Quadratic Trading Example

Linear quadratic trading: Dynamics

- $x_{t+1} = f_t(x_t, u_t, \rho_t) = \mathbf{diag}(\rho_t)(x_t + u_t)$
- $x_t \in \mathbb{R}^n$ is dollar amount of holding in n assets
- $(x_t)_i < 0$ means short position in asset i in period t
- $u_t \in \mathbb{R}^n$ is dollar amount of each asset bought at beginning of period t
- $(u_t)_i < 0$ means asset i is sold in period t
- $x_t^+ = x_t + u_t$ is post-trade portfolio
- $\rho_t \in \mathbb{R}^n_{++}$ is (random) return of assets over period $(t, t+1]$
- returns independent, with $\mathbf{E}\,\rho_t = \overline{\rho}_t$, $\mathbf{E}\,\rho_t \rho_t^T = \Sigma_t$

Linear quadratic trading: Stage cost

stage cost for $t = 0, \ldots, T-1$ is (convex quadratic)

$$g_t(x, u) = \mathbf{1}^T u + \frac{1}{2}(\kappa_t^T u^2 + \gamma (x+u)^T Q_t (x+u))$$

with $Q_t > 0$

- first term is gross cash in
- second term is quadratic transaction cost (square is elementwise; $\kappa_t > 0$)
- third term is risk (variance of post-trade portfolio for $Q_t = \Sigma_t - \overline{\rho}_t \overline{\rho}_t^T$)
- $\gamma > 0$ is risk aversion parameter
- minimizing total stage cost equivalent to maximizing (risk-penalized) net cash taken from portfolio

Linear quadratic trading: Terminal cost

- terminal cost: $g_T(x) = -\mathbf{1}^\mathsf{T} x + \frac{1}{2}\kappa_T^\mathsf{T} x^2$, $\kappa_T > 0$
- this is net cash in if we close out (liquidate) final positions, with quadratic transaction cost

Linear quadratic trading: DP

- value functions quadratic (including linear and constant terms):

$$v_t(x) = \frac{1}{2}(x^T P_t x + 2q_t^T x + r_t)$$

- we'll need formula

$$\mathbf{E}(\mathbf{diag}(\rho_t) P \, \mathbf{diag}(\rho_t)) = P \circ \Sigma_t$$

  where $\circ$ is Hadamard (element-wise) product

- optimal expected tail cost

$$\begin{aligned} \mathbf{E}\, v_{t+1}(f_t(x,u,\rho_t)) &= \mathbf{E}\, v_{t+1}(\mathbf{diag}(\rho_t) x^+) \\ &= \tfrac{1}{2}((x^+)^T P_{t+1} \circ \Sigma_t x^+ + 2q_{t+1}^T \, \mathbf{diag}(\overline{\rho}_t) x^+ + r_{t+1}) \end{aligned}$$

Linear quadratic trading: DP

- $P_T = \mathbf{diag}(\kappa_T)$, $q_T = -\mathbf{1}$, $r_T = 0$
- recall $v_t(x) = \min_u \mathbf{E}\left(g_t(x,u) + v_{t+1}(\mathbf{diag}(\rho_t)(x+u))\right)$
- for $t = T-1, \ldots, 0$ we minimize over u to get optimal policy:

$$\begin{aligned}\mu_t(x) &= \operatorname{argmin}_u \left(u^T(S_{t+1} + \mathbf{diag}(\kappa_t))u + 2(S_{t+1}x + s_{t+1} + \mathbf{1})^T u\right) \\ &= -(S_{t+1} + \mathbf{diag}(\kappa_t))^{-1}(S_{t+1}x + s_{t+1} + \mathbf{1}) \\ &= K_t x + l_t\end{aligned}$$

  where

$$S_{t+1} = P_{t+1} \circ \Sigma_t + \gamma Q_t, \qquad s_{t+1} = \overline{\rho}_t \circ q_{t+1}$$

- using $u = K_t x + l_t$ we then have

$$v_t(x) = \frac{1}{2}\left[\begin{array}{c} x \\ 1 \end{array}\right]^T \left[\begin{array}{cc} S_{t+1}(I + K_t) & s_{t+1} + S_{t+1} l_t \\ s_{t+1}^T + l_t^T S_{t+1} & r_{t+1} + (s_{t+1} + \mathbf{1})^T l_t \end{array}\right]\left[\begin{array}{c} x \\ 1 \end{array}\right]$$

Linear quadratic trading: value iteration

- set $P_T = \mathbf{diag}(\kappa_T)$, $q_T = -\mathbf{1}$, $r_T = 0$
- for $t = T-1, \ldots, 0$

$$\begin{array}{rcl} K_t &=& -(S_{t+1} + \mathbf{diag}(\kappa_t))^{-1} S_{t+1} \\ l_t &=& -(S_{t+1} + \mathbf{diag}(\kappa_t))^{-1}(s_{t+1} + \mathbf{1}) \\ P_t &=& S_{t+1}(I + K_t) \\ q_t &=& s_{t+1} + S_{t+1} l_t \\ r_t &=& r_{t+1} + (s_{t+1} + \mathbf{1})^\mathsf{T} l_t \end{array}$$

  where

$$S_{t+1} = P_{t+1} \circ \Sigma_t + \gamma Q_t, \qquad s_{t+1} = \overline{\rho}_t \circ q_{t+1}$$

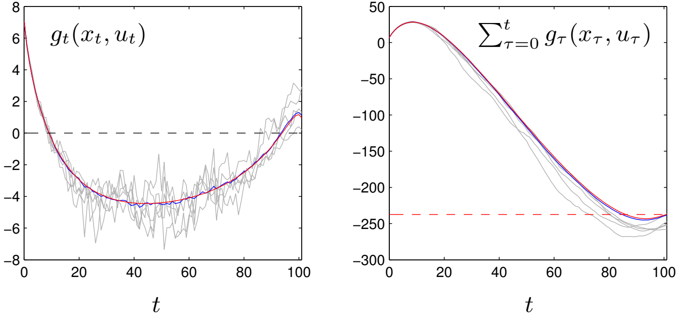
- optimal policy: $\mu_t^\star(x) = K_t x + l_t$
- can write as $\mu_t^\star(x) = K_t(x - x_t^{\mathrm{tar}})$, $\quad x_t^{\mathrm{tar}} = -K_t^{-1} l_t = -S_{t+1}^{-1}(s_{t+1} + \mathbf{1})$
- $J^\star = \mathbf{E}\, v_0(x_0)$

Linear quadratic trading: Numerical instance

- $n = 30$ assets over $T = 100$ time-steps
- initial portfolio $x_0 = 0$
- $\overline{\rho}_t = \overline{\rho}$, $\Sigma_t = \Sigma$ for $t = 0, \dots, T-1$
- $Q_t = \Sigma - \overline{\rho\rho}^\mathsf{T}$ for $t = 0, \dots, T-1$
- asset returns log-normal, expected returns range over $\pm 3\%$ per period
- asset return standard deviations range from 0.4% to 9.8%
- asset correlations range from -0.3 to 0.8

Linear quadratic trading: Numerical instance

- $N = 100$ Monte Carlo simulations
- $J^\star = v_0(x_0) = -237.5$ (Monte Carlo estimate: -238.4)
- exact (red), MC estimate (blue), and samples (gray); $J^\star$ red dashed



Linear quadratic trading: Numerical instance

we define $x_{T+1} = 0$, *i.e.*, we close out the position during period T