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The linear model

A very important class of estimation problems is the *linear model*

$$y = Ax + w$$

- x and w are independent
- We have induced pdfs p^x for x and p^w for w
- The matrix A is $m \times n$

We measure $y = y_{\text{meas}}$ and would like to estimate x

The mean

- Let $\mu_x = \mathbf{E}\, x$ and $\mu_w = \mathbf{E}\, w$

- Then

$$\boxed{\mathbf{E}\, y = A\mu_x + \mu_w}$$

- Call this μ_y

The linear map

Since $y = Ax + w$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

- We will measure $y = y_{\text{meas}}$ and estimate x
- To do this, we would like the *conditional pdf* of $x \mid y = y_{\text{meas}}$
- For this, we need the joint pdf of x and y

The joint pdf

The joint pdf p of x and y is

$$\boxed{p(x,y) = p^x(x)p^w(y - Ax)}$$

because the joint pdf of x, w is

$$p_1\left(\begin{bmatrix} x \\ w \end{bmatrix}\right) = p^x(x)p^w(w)$$

and we know

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

$z = Hu$ implies $p^z(a) = |\det H|^{-1} p^u(H^{-1}(u))$, so

$$p(x,y) = p_1\left(\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

The covariance

Let $\Sigma_w = \mathbf{cov}(w)$ and $\Sigma_x = \mathbf{cov}(x)$. We have

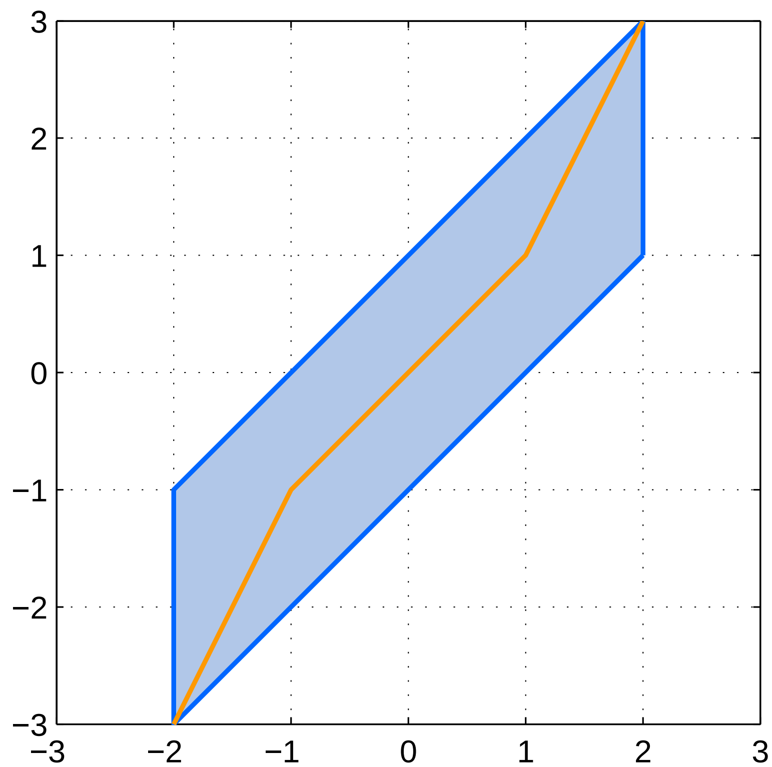
$$\boxed{\mathbf{cov}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Sigma_x & \Sigma_x A^T \\ A\Sigma_x & A\Sigma_x A^T + \Sigma_w \end{bmatrix}}$$

- Call this $\begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix}$. Above holds because $\mathbf{cov}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_w \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix}^T$
- Then $\mathbf{cov}(y) = A\Sigma_x A^T + \Sigma_w$
- $A\Sigma_x A^T$ is 'signal covariance'
- Σ_w is 'noise covariance'

Example: uniform pdfs

Suppose $x \sim \mathcal{U}[-2, 2]$ and $w \sim \mathcal{U}[-1, 1]$ and we measure $y = x + w$.

The joint pdf and MMSE estimator are

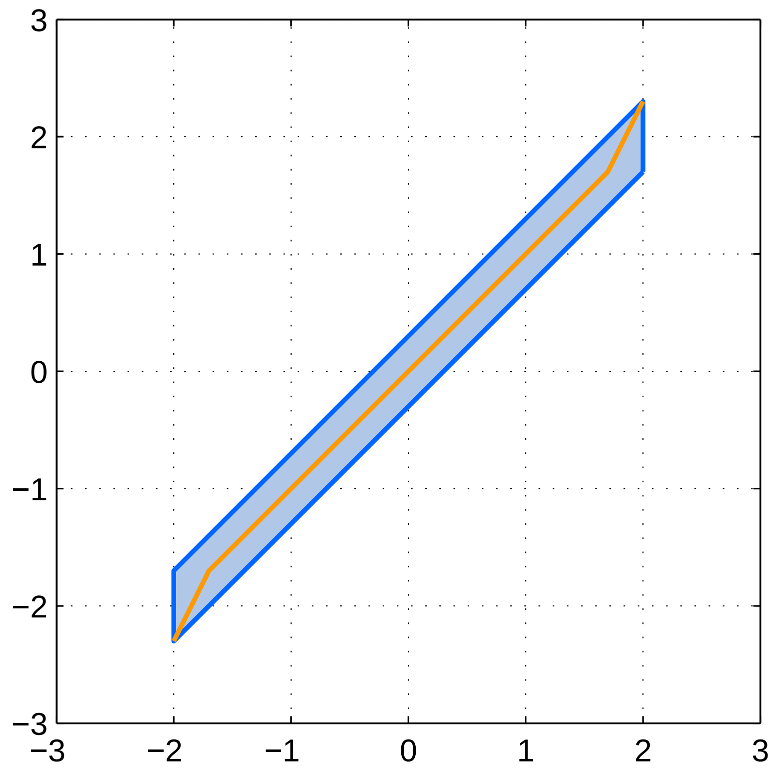


Notice the estimator is not $x_{\text{est}} = y_{\text{meas}}$, because of the prior information that $x \in [-2, 2]$.

The importance of the prior

- $x \sim \mathcal{U}[-2, 2]$ as before
- $w \sim \mathcal{U}[-0.3, 0.3]$; signal x is large relative to the noise w

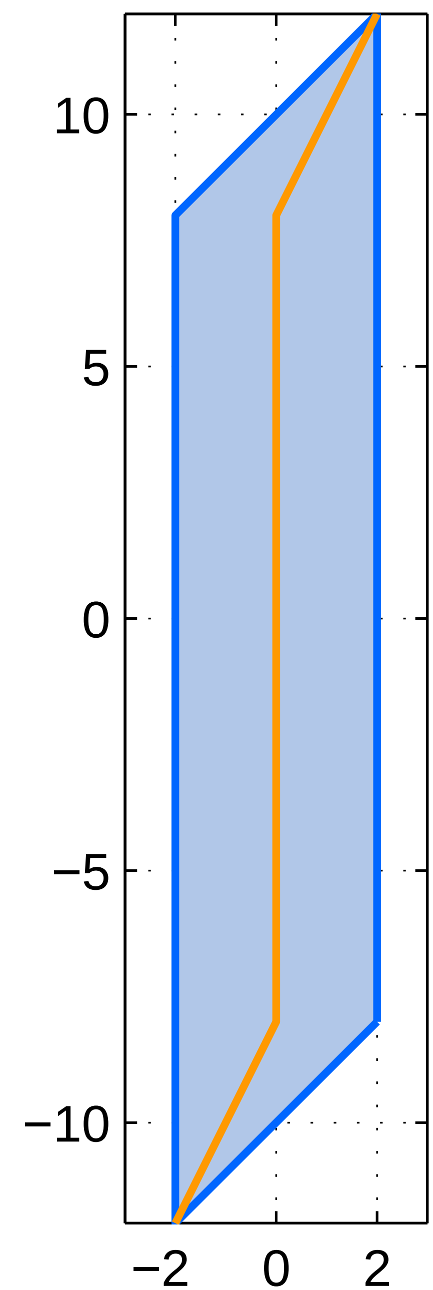
The joint pdf and MMSE estimator are



The estimator is *almost* $x_{\text{est}} = y_{\text{meas}}$

The importance of the prior

- $x \sim \mathcal{U}[-2, 2]$ as before
- $w \sim \mathcal{U}[-10, 10]$; signal x is small relative to the noise w



- The joint pdf and MMSE estimator are shown.
- The estimator mostly ignores the measurement
- The estimate is *almost* the *prior mean* $\mathbf{E}\, x = 0$

Linear measurements with Gaussian noise

We have the *linear model*

$$y = Ax + w$$

- $x \sim \mathcal{N}(0, \Sigma_x)$ and $w \sim \mathcal{N}(0, \Sigma_w)$ are independent
- So $\begin{bmatrix} x \\ y \end{bmatrix}$ is Gaussian, with mean and covariance

$$\mathbf{E} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbf{cov} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Sigma_x & \Sigma_x A^T \\ A\Sigma_x & A\Sigma_x A^T + \Sigma_w \end{bmatrix}$$

Linear measurements with Gaussian noise

The MMSE estimate of x given $y = y_{\text{meas}}$ is

$$\boxed{\hat{x}_{\text{mmse}} = \Sigma_x A^T (A \Sigma_x A^T + \Sigma_w)^{-1} y_{\text{meas}}}$$

because we know $\hat{x}_{\text{mmse}} = \Sigma_{xy} \Sigma_y^{-1} y_{\text{meas}}$

The matrix $L = \Sigma_x A^T (A \Sigma_x A^T + \Sigma_w)^{-1}$ is called the *estimator gain*

Example: linear measurements with Gaussian noise

Suppose $y = 2x + w$, with

- prior covariance $\mathbf{cov}(x) = 1$
- noise covariance $\mathbf{cov}(w) = 3$

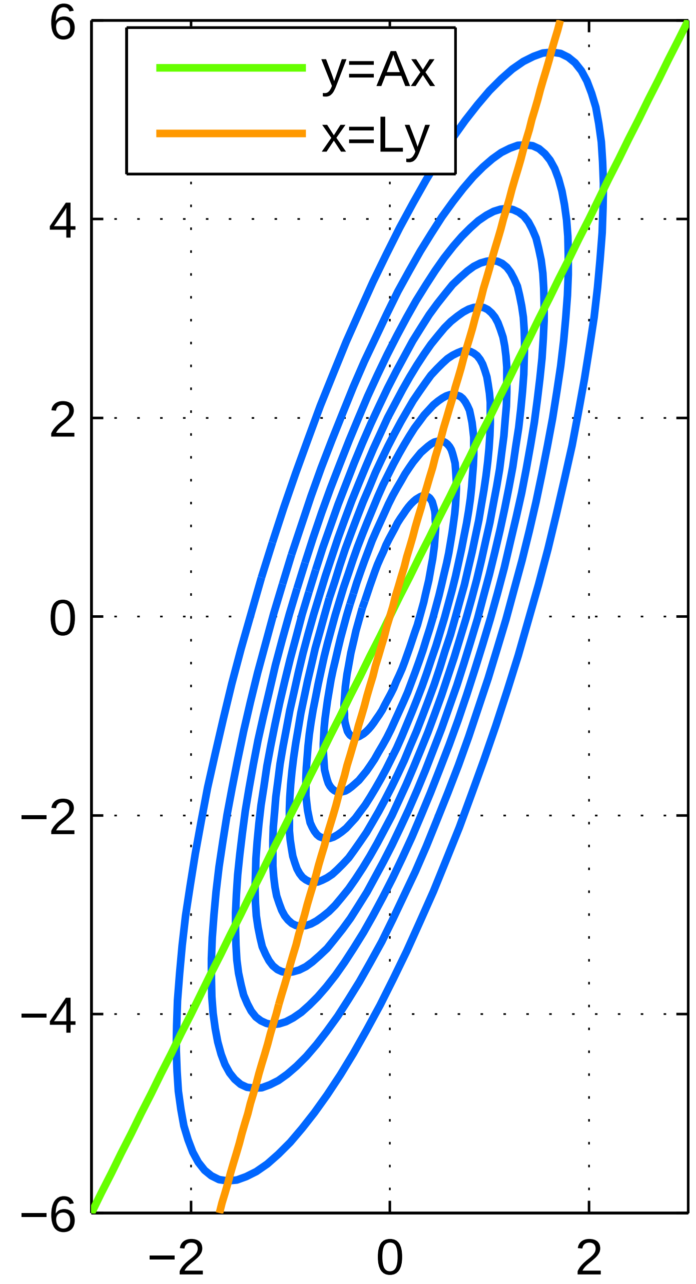
the estimator is

$$x_{\text{mmse}} = \frac{2y_{\text{meas}}}{7}$$

The MMSE estimator gives a smaller answer than just inverting A,

$$|x_{\text{mmse}}| \le |A^{-1}y_{\text{meas}}|$$

since we have prior information about x



Non-zero means

Suppose $x \sim \mathcal{N}(\mu_x, \Sigma_x)$ and $w \sim \mathcal{N}(\mu_w, \Sigma_w)$.

The MMSE estimate of x given $y = y_{\text{meas}}$ is

$$\boxed{\hat{x}_{\text{mmse}} = \mu_x + \Sigma_x A^T (A \Sigma_x A^T + \Sigma_w)^{-1} (y_{\text{meas}} - A\mu_x - \mu_w)}$$

The signal to noise ratio

Suppose where x, y and w are scalar, and $y = Ax + w$. The *signal-to-noise ratio* is

$$\boxed{s = \frac{\sqrt{A^2\Sigma_x}}{\sqrt{\Sigma_w}}}$$

- Commonly used for scalar w, x, y; no use in vector case
- In terms of s, the MMSE estimate is

$$x_{\text{mmse}} = \mu_x + \frac{A\Sigma_x}{A^2\Sigma_x + \Sigma_w}(y_{\text{meas}} - A\mu_x)$$

$$= \frac{1}{1+s^2}\mu_x \; + \; \frac{s^2}{1+s^2}A^{-1}y_{\text{meas}}$$

Scalar systems and the SNR

The MMSE estimate is

$$\boxed{x_{\text{mmse}} = \frac{1}{1+s^2}\mu_x \; + \; \frac{s^2}{1+s^2}A^{-1}y_{\text{meas}}}$$

- let $\theta = \dfrac{1}{1+s^2}$, then $x_{\text{mmse}} = \theta\mu_x + (1-\theta)A^{-1}y$

  a *convex linear combination* of the prior mean and the least-squares estimate

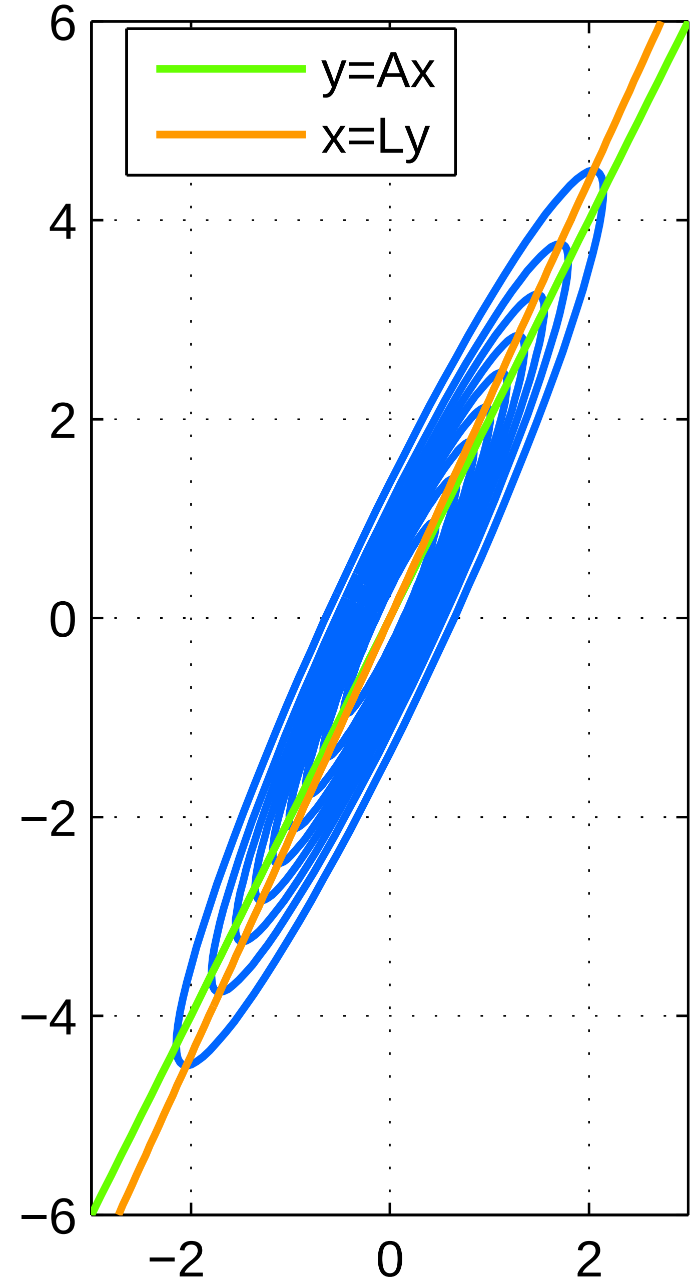
- when s is small, $x_{\text{mmse}} \approx \mu_x$, the *prior mean*

- when s is large, $x_{\text{mmse}} \approx A^{-1}y$, the *least-squares estimate* of y

Example: small noise

Suppose $y = 2x + w$, with

- prior covariance $\mathbf{cov}(x) = 1$
- noise covariance $\mathbf{cov}(w) = 0.4$; signal is large compared to noise



Hence

- SNR $s = \dfrac{\sqrt{A^2\Sigma_x}}{\sqrt{\Sigma_w}} \approx 3.2$
- Estimate is

$$\begin{aligned} x_{\text{mmse}} &= \frac{s^2}{1+s^2} A^{-1} y_{\text{meas}} \\ &\approx 0.9 A^{-1} y_{\text{meas}} \end{aligned}$$

i.e., close to $y_{\text{meas}}/2$

Example: large noise

Suppose $y = 2x + w$, with

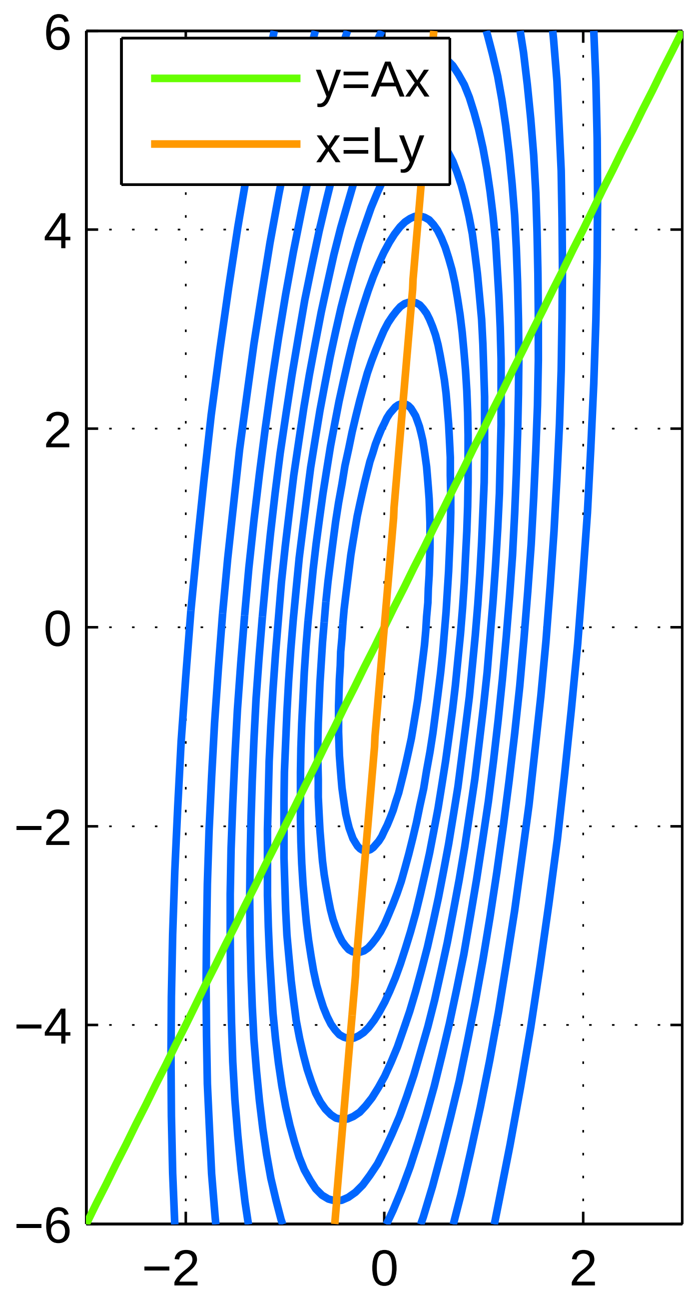
- prior covariance $\mathbf{cov}(x) = 1$
- noise covariance $\mathbf{cov}(w) = 20$; signal is small compared to noise

Hence

- SNR $s = \dfrac{\sqrt{A^2\Sigma_x}}{\sqrt{\Sigma_w}} \approx 0.45$
- Estimate is

$$
\begin{aligned}
x_{\text{mmse}} &= \frac{s^2}{1+s^2} A^{-1} y_{\text{meas}} \\
&\approx 0.17 A^{-1} y_{\text{meas}}
\end{aligned}
$$

i.e., closer to 0 for all y_{meas}



The posterior covariance

The posterior covariance of x given $y = y_{\text{meas}}$ is

$$\boxed{\mathbf{cov}\big(x \,\big|\, y = y_{\text{meas}}\big) = \Sigma_x - \Sigma_x A^T (A\Sigma_x A^T + \Sigma_w)^{-1} A\Sigma_x}$$

- above follows because

$$\mathbf{cov}\big(x \,\big|\, y = y_{\text{meas}}\big) = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}$$

- We can use this to compute the MSE since

$$\mathbf{E}\big(\|x - \hat{x}_{\text{mmse}}\|^2 \,\big|\, y = y_{\text{meas}}\big) = \mathbf{trace}\,\mathbf{cov}\big(x \,\big|\, y = y_{\text{meas}}\big)$$

The posterior covariance and SNR

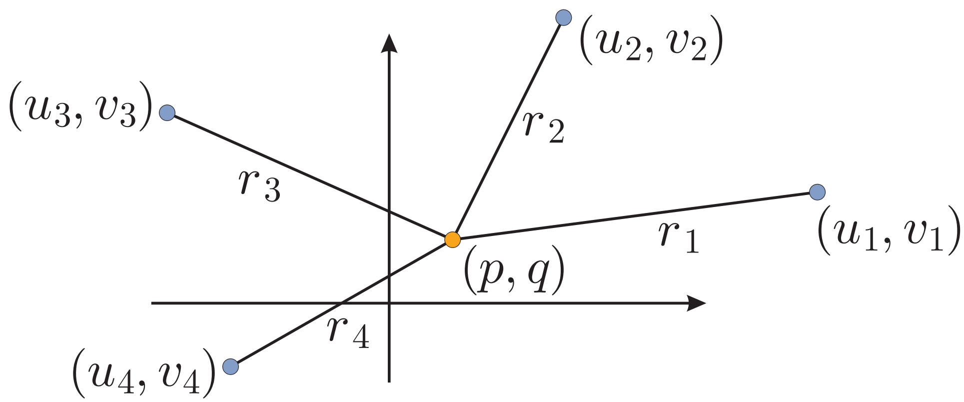
For scalar problems, the posterior covariance of x given $y = y_{\text{meas}}$ is

$$\boxed{\mathbf{cov}\big(x \,\big|\, y = y_{\text{meas}}\big) = \frac{\Sigma_x}{1 + s^2}}$$

- The *uncertainty* (covariance) in x is reduced by the factor $\dfrac{1}{1+s^2}$ by measurement

Example: navigation

$x = \begin{bmatrix} p \\ q \end{bmatrix}$ our location, we measure distances r_i to m beacons at points (u_i, v_i)



assume p, q are small compared to u_i, v_i. then, approximately

$$\boxed{y = Ax}$$

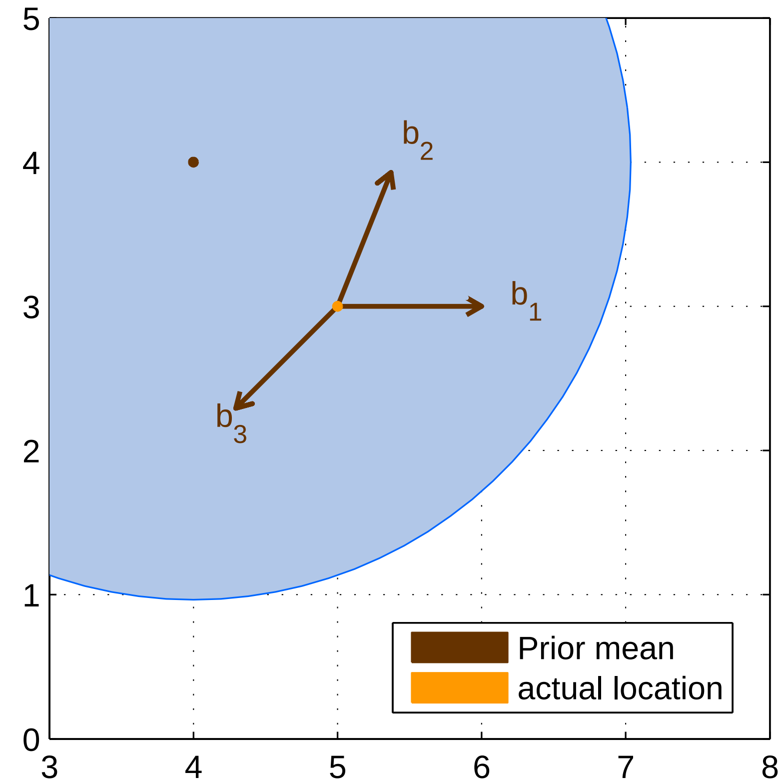
- $A \in \mathbb{R}^{m \times 2}$, ith row of A is the transpose of unit vector in the direction of beacon i
- $y = \begin{bmatrix} \sqrt{u_1^2 + v_1^2} - r_1 \\ \vdots \\ \sqrt{u_m^2 + v_m^2} - r_m \end{bmatrix}$ measured vector of distances

Example: navigation

here $A \in \mathbb{R}^{3\times 2}$ with

$$A = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

and $y = Ax$. Each b_i is a unit vector.

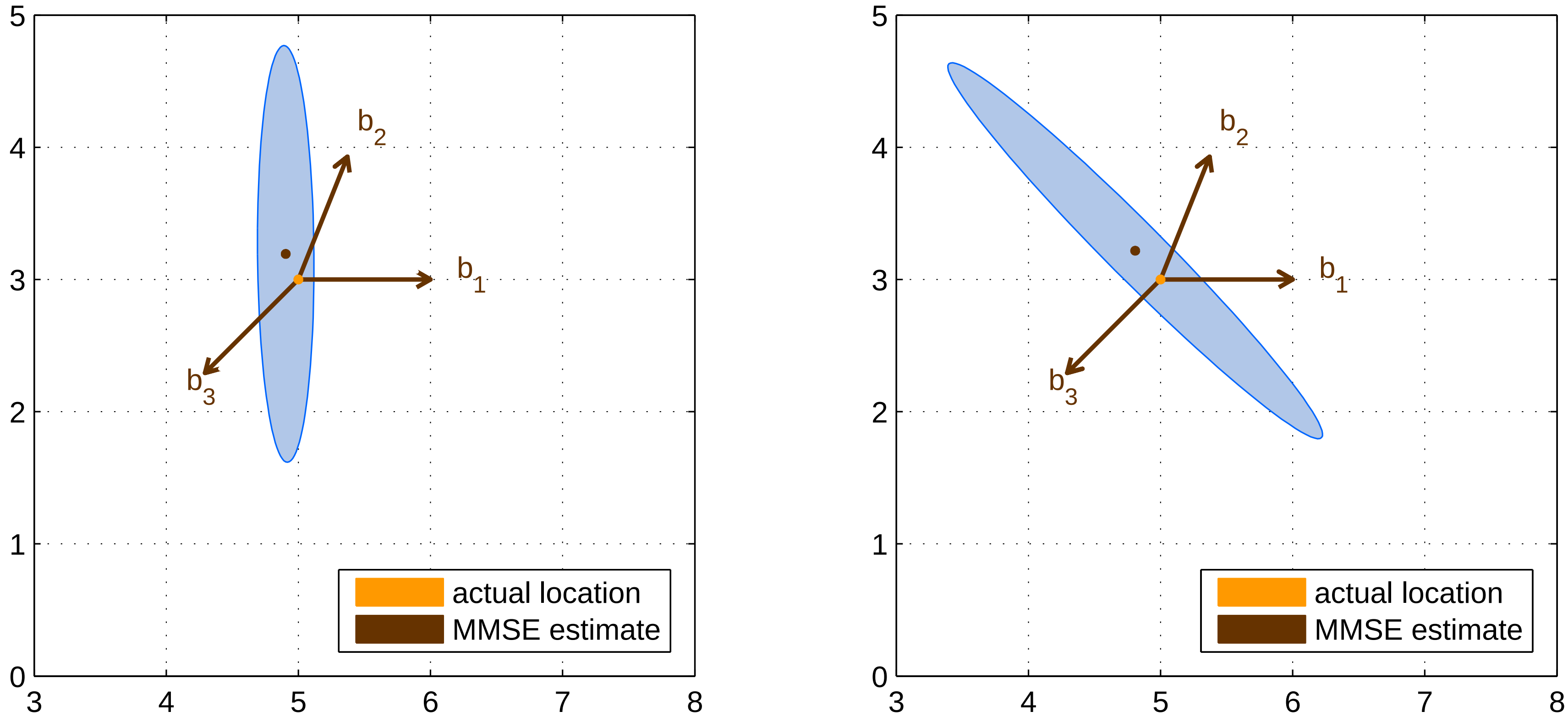


- Prior information is $x \sim \mathcal{N}\left(\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$
- y is measured; y_i is range measurement in the direction b_i with noise w added
- beacons at $\begin{bmatrix} 50 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ 50 \end{bmatrix}, \begin{bmatrix} -50 \\ -50 \end{bmatrix}$
- figure shows prior 90% confidence ellipsoid

Example: posterior confidence ellipsoids

Posterior confidence ellipsoids for two different possible noise covariances.

$$\Sigma_w = \begin{bmatrix} 0.01 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \Sigma_w = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.01 \end{bmatrix}$$



Alternative formula

There is another way to write the posterior covariance:

$$\boxed{\mathbf{cov}\big(x \,|\, y = y_{\mathsf{meas}}\big) = \big(\Sigma_x^{-1} + A^T \Sigma_w^{-1} A\big)^{-1}}$$

- follows from the *Sherman-Morrison-Woodbury* formula

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}$$

- This is very useful when we have fewer unknowns than measurements; i.e., Σ_x is smaller that $A\Sigma_x A^T$

Alternative formula

There is also an alternative formula for the estimator gain

$$\boxed{L = (\Sigma_x^{-1} + A^T \Sigma_w^{-1} A)^{-1} A^T \Sigma_w^{-1}}$$

- Because

$$\begin{aligned} L &= \Sigma_x A^T (A \Sigma_x A^T + \Sigma_w)^{-1} \\ &= \Sigma_x A^T (\Sigma_w^{-1} A \Sigma_x A^T + I)^{-1} \Sigma_w^{-1} \\ &= (\Sigma_x A^T \Sigma_w^{-1} A + I)^{-1} \Sigma_x A^T \Sigma_w^{-1} \qquad \text{by push-through identity} \\ &= (A^T \Sigma_w^{-1} A + \Sigma_x^{-1})^{-1} A^T \Sigma_w^{-1} \end{aligned}$$

Comparison with least-squares

The least-squares approach minimizes

$$\|y - Ax\|^2 = \sum_{i=1}^{m} \bigl(y_i - a_i^T x\bigr)^2$$

where $A = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}^T$

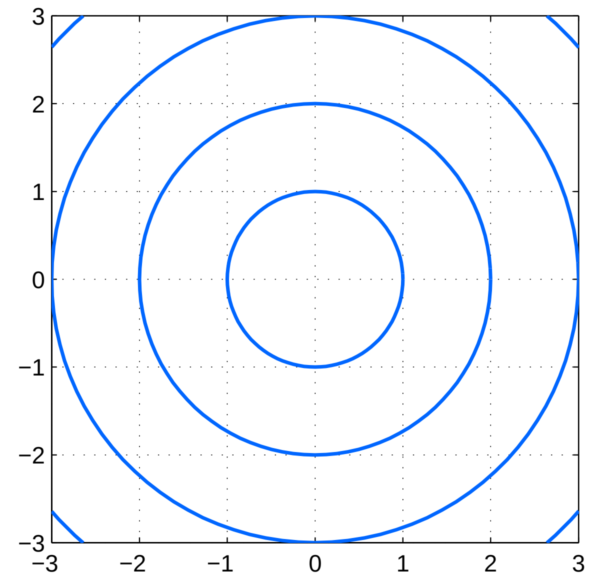
Suppose instead we minimize

$$\sum_{i=1}^{m} w_i \bigl(y_i - a_i^T x\bigr)^2$$

where $w_1, w_2, \dots, w_m$ are positive *weights*

Weighted norms

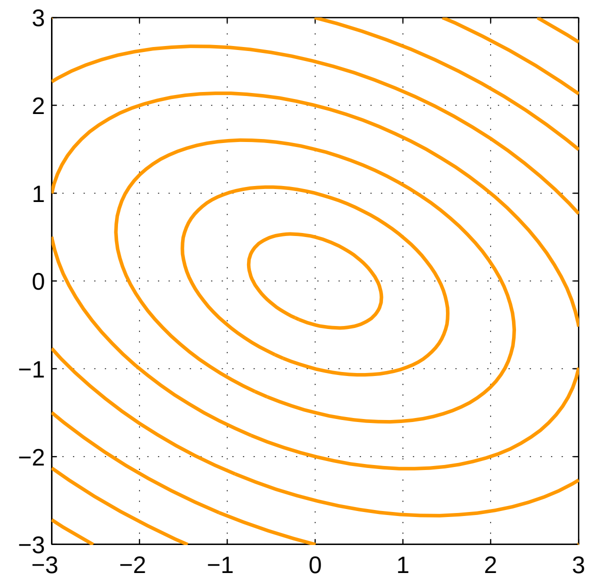
More generally, let's look at *weighted norms*



contours of the 2-norm

$$\|x\|_2 = \sqrt{x^T x}$$

contours of the *weighted-norm*



$$\|x\|_W = \sqrt{x^T W x}$$

$$= \|W^{\frac{1}{2}} x\|_2$$

where $W = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

Weighted least squares

the *weighted least-squares* problem; given $y_{\text{meas}} \in \mathbb{R}^m$,

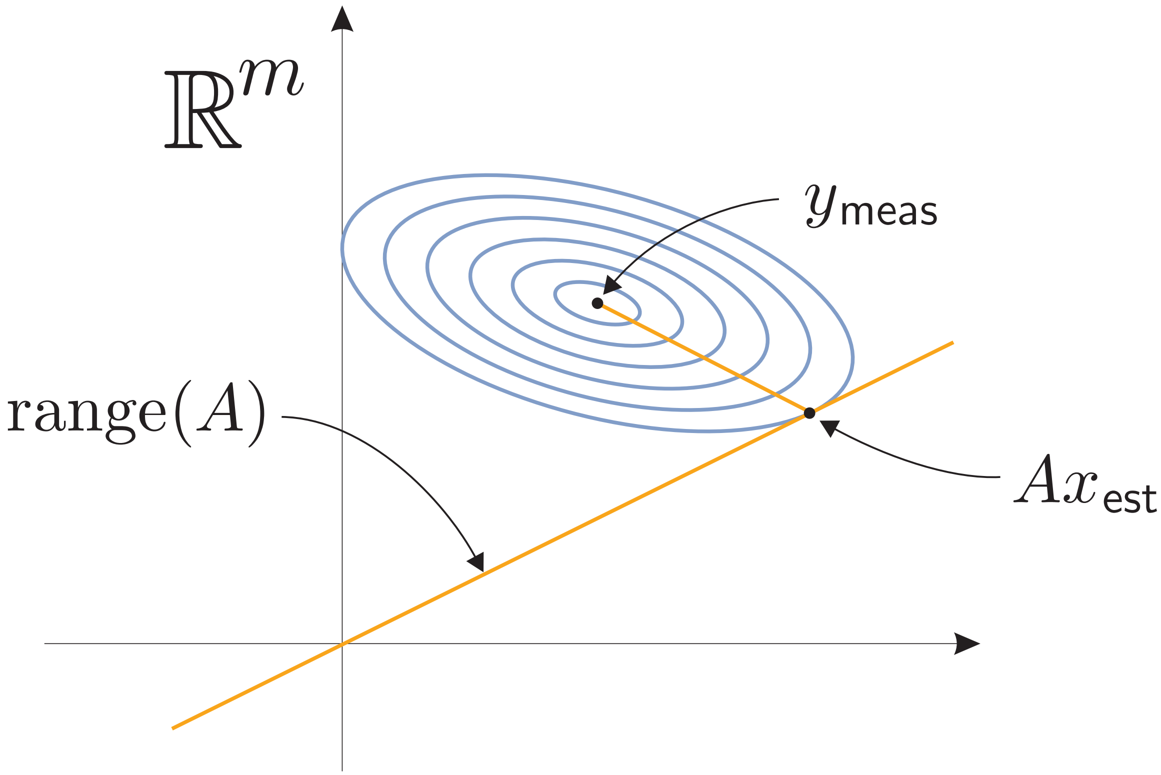
$$\boxed{\text{minimize} \qquad \|y_{\text{meas}} - Ax\|_W}$$

assume $A \in \mathbb{R}^{m \times n}$, skinny, full rank, and $W \in \mathbb{R}^{m \times m}$ and $W > 0$

then (by differentiating) the optimum x is

$$\boxed{x_{\text{wls}} = (A^T W A)^{-1} A^T W y_{\text{meas}}}$$

Weighted least squares



- if there is no noise, y lies in $\mathbf{range}\, A$
- the weighted least-squares estimate x_{wls} minimizes

$$\|y_{\text{meas}} - Ax\|_W$$

- Ax_{wls} is the closest (in weighted-norm) point in $\mathbf{range}\, A$ to y_{meas}

MMSE and weighted least squares

suppose we choose weight $W = \Sigma_w^{-1}$; then WLS solution is

$$x_{\text{wls}} = (A^T \Sigma_w^{-1} A)^{-1} A^T \Sigma_w^{-1} y_{\text{meas}}$$

compare with MMSE estimate when $x \sim \mathcal{N}(0, \Sigma_x)$ and $w \sim \mathcal{N}(0, \Sigma_w)$

$$x_{\text{mmse}} = (\Sigma_x^{-1} + A^T \Sigma_w^{-1} A)^{-1} A^T \Sigma_w^{-1} y_{\text{meas}}$$

- as the prior covariance $\Sigma_x \to \infty$, the MMSE estimate tends to the WLS estimate
- if $\Sigma_w = I$ then MMSE tends to usual least-squares solution as $\Sigma_x \to \infty$
- the weighted norm heavily penalizes the residual $y - Ax$ in low-noise directions