# 10 Factorization

## 10.1 State-Space Factorization

We have the simplest factorization result below. First, recall the product realization

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

**Lemma 1.** *Suppose*

$$G = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \qquad H_1 = \left[\begin{array}{c|c} A_{22} & B_2D_2^{-1} \\ \hline C_2 & D_1 \end{array}\right] \qquad H_2 = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline D_1^{-1}C_1 & D_2 \end{array}\right]$$

*where $D$ is invertible and $D = D_1D_2$. Let $A^\times$ be the generator for $G^{-1}$ in the same coordinates as $G$, that is $A^\times = A - BD^{-1}C$. If*

$$A_{12} = 0 \qquad \textit{and} \qquad A^\times_{21} = 0$$

*then*

$$G = H_1H_2$$

**Proof.** This is immediate, since

$$A^\times = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} D^{-1} \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and

$$H_1H_2 = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ B_2D^{-1}C_1 & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

■

Notice in particular that this means that if

$$\text{range} \begin{bmatrix} 0 \\ I \end{bmatrix} \text{ is } A\text{-invariant}$$

and

$$\text{range} \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ is } A^\times\text{-invariant}$$

then it is easy to factorize $G$. We also have

$$H_1^{-1} = \left[\begin{array}{c|c} A_{22} - B_2D^{-1}C_2 & -B_2D^{-1} \\ \hline D_1^{-1}C_2 & D_1^{-1} \end{array}\right] \qquad H_2^{-1} = \left[\begin{array}{c|c} A_{11} - B_1D^{-1}C_1 & -B_1D_2^{-1} \\ \hline D^{-1}C_1 & D_2^{-1} \end{array}\right]$$

Another version of Lemma 1 is below.

**Lemma 2.** *Suppose*

$$G = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \qquad H_1 = \left[\begin{array}{c|c} A_{11} & B_1 D_2^{-1} \\ \hline C_1 & D_1 \end{array}\right] \qquad H_2 = \left[\begin{array}{c|c} A_{22} & B_2 \\ \hline D_1^{-1} C_2 & D_2 \end{array}\right]$$

*and $D = D_1 D_2$. Let $A^\times$ be the generator for $G^{-1}$, that is $A^\times = A - BD^{-1}C$. If $A_{21} = 0$ and $A^\times_{12} = 0$ then $G = H_1 H_2$.*

More generally, we would like to change coordinates to achieve the conditions required by Lemma 1, that is $A_{12} = 0$ and $A^\times_{21} = 0$. In other words, we need coordinates in which $A^\times$ is upper triangular, and this corresponds to finding a subspace $V$ which is $A^\times$ invariant. We also need those coordinates to be such that $A$ is lower triangular.

Suppose $A_{11}$ and $A_{22}$ are $n \times n$. Then we need $V$ to be at least $n$ dimensional, since we need $A^\times_{21} = 0$. Suppose $V$ is given by

$$V = \operatorname{range} T_1$$

We will use a change of coordinates

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$$

then under this coordinate change $A^\times$ will be upper triangular if $V$ is $A^\times$-invariant.

However, we also need $A$ to be lower triangular in these new coordinates, and so we need $\operatorname{range} T_2$ to be $A$-invariant. One choice of $T_2$ which will satisfy this is

$$T_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

since $A$ is lower triangular. These two subspaces must be complementary, that is

$$\operatorname{range} T_1 \oplus \operatorname{range} \begin{bmatrix} 0 \\ I \end{bmatrix} = \mathbb{R}^{2n}$$

and this holds if and only if $T$ will be invertible. Let

$$T_1 = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

Then $T$ is invertible if and only if $P_1$ is invertible. Then

$$\operatorname{range} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} I \\ P_2 P_1^{-1} \end{bmatrix}$$

So we choose

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

This motivates the following problem.

**Lemma 3.** *Suppose $V \subset \mathbb{R}^n$ is an $n$-dimensional subspace. Then*

$$V \oplus \operatorname{range}\begin{bmatrix}0\\ I\end{bmatrix} = \mathbb{R}^{2n} \qquad \text{and } V \text{ is } A^\times\text{-invariant}$$

*if and only if there exists $P$ such that*

$$V = \operatorname{range}\begin{bmatrix}I\\ P\end{bmatrix}$$

*and*

$$\begin{bmatrix}I & 0\\ -P & I\end{bmatrix} A^\times \begin{bmatrix}I & 0\\ P & I\end{bmatrix} \text{ is block lower triangular}$$

Now let

$$A^\times = \begin{bmatrix}A^\times_{11} & A^\times_{12}\\ A^\times_{21} & A^\times_{22}\end{bmatrix} \qquad T = \begin{bmatrix}I & 0\\ P & I\end{bmatrix}$$

Then

$$T^{-1}A^\times T = \begin{bmatrix}A^\times_{11} + A^\times_{12}P & A^\times_{12}\\ -PA^\times_{11} + A^\times_{21} - PA^\times_{12}P + A^\times_{22}P & -PA^\times_{12} + A^\times_{22}\end{bmatrix}$$

and then $A^\times_{21} = 0$ if

$$-PA^\times_{11} + A^\times_{21} - PA^\times_{12}P + A^\times_{22}P = 0$$

This equation is called the ***Riccati*** equation. Note that we can also write the Riccati equation as

$$\begin{bmatrix}-P & I\end{bmatrix} A^\times \begin{bmatrix}I\\ P\end{bmatrix} = 0$$

Let's summarize so far.

**Theorem 4.** *Suppose*

$$G = \left[\begin{array}{cc|c} A_{11} & 0 & B_1\\ A_{21} & A_{22} & B_2\\ \hline C_1 & C_2 & D\end{array}\right]$$

*where $D$ is invertible. Let $A^\times = A - BD^{-1}C$, and suppose $P$ satisfies the Riccati equation*

$$\begin{bmatrix}-P & I\end{bmatrix} A^\times \begin{bmatrix}I\\ P\end{bmatrix} = 0$$

*Let $D_1 D_2 = D$ and*

$$H_1 = \left[\begin{array}{c|c} A_{22} & (-PB_1 + B_2)D_2^{-1}\\ \hline C_2 & D_1\end{array}\right] \qquad H_2 = \left[\begin{array}{c|c} A_{11} & B_1\\ \hline D_1^{-1}(C_1 + C_2P) & D_2\end{array}\right]$$

*Then $G = H_1 H_2$.*

**Proof.** To prove this, we use coordinates

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

then

$$G = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ -PA_{11} + A_{21} + A_{22}P & A_{22} & -PB_1 + B_2 \\ \hline C_1 + C_2 P & C_2 & D \end{array}\right]$$

and

$$\begin{aligned} T^{-1}A^\times T &= \begin{bmatrix} A_{11}^\times + A_{12}^\times P & A_{12}^\times \\ -PA_{11}^\times + A_{21}^\times - PA_{12}^\times P + A_{22}^\times P & -PA_{12}^\times + A_{22}^\times \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^\times + A_{12}^\times P & A_{12}^\times \\ 0 & -PA_{12}^\times + A_{22}^\times \end{bmatrix} \end{aligned}$$

Then Lemma 1 gives the desired result. ∎

## 10.2 Spectral Factorization

We'll need the following expressions for rational functions. This is simply algebraic manipulation.

**Lemma 5.** *Suppose $G(z) = C(zI - A)^{-1}B + D$ and $A$ is invertible. Then*

$$\begin{aligned} \tilde{G}(z) &= D^T + B^T(z^{-1}I - A)^{-1}C^T \\ &= \left[\begin{array}{c|c} A^{-T} & -A^{-T}C^T \\ \hline B^T A^{-T} & D^T - B^T A^{-T} C^T \end{array}\right](z) \end{aligned}$$

*and*

$$\tilde{G}G = \left[\begin{array}{cc|c} A & 0 & B \\ -A^{-T}C^T C & A^{-T} & -A^{-T}C^T D \\ \hline D^T C - B^T A^{-T} C^T C & B^T A^{-T} & D^T D - B^T A^{-T} C^T D \end{array}\right]$$

**Proof.** By definition

$$\tilde{G}(z) = D^T + B^T(z^{-1}I - A)^{-1}C^T$$

and we need to express the rational $(z^{-1}I - A)^{-1}$ in terms of $(zI - Q)^{-1}$ for some $Q$. We have

$$z(zI - A)^{-1} = I + (zI - A)^{-1}A$$

and hence

$$(z^{-1}I - A^T)^{-1} = -A^{-T}\big(I + (zI - A^{-T})^{-1}A^{-T}\big)$$

This immediately gives the first result. The second result follows from the usual product expression. ∎

We'd like to use our previous factorization results to construct the spectral factorization of $\tilde{G}G$. Assume for simplicity that $C^T D = 0$ and $D^T D > 0$, and let $H$ be the generator matrix for a realization of $(\tilde{G}G)^{-1}$ so that

$$H = \begin{bmatrix} A & 0 \\ -A^{-T}C^T C & A^{-T} \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} (D^T D)^{-1} \begin{bmatrix} -B^T A^{-T} C^T C & B^T A^{-T} \end{bmatrix}$$

Define for convenience

$$Q = C^T C \qquad R = D^T D \qquad G = BR^{-1}B^T$$

then

$$H = \begin{bmatrix} A + GA^{-T}Q & -GA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \tag{1}$$

Also define

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

Then the matrix $H$ satisfies

$$J^{-1} H^T J = J^{-1}$$

Such matrices are called *symplectic*. This implies that $H$ is similar to $H^{-T}$, and therefore if $\lambda$ is an eigenvalue of $H$ so is $1/\bar{\lambda}$. One can show that every symplectic matrix has the form of equation (1).

### 10.2.1 The Riccati Equation

We'd like to apply Theorem 4 to factorize $\tilde{G}G$. The corresponding Riccati equation is

$$\begin{bmatrix} -P & I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

which is just

$$-PA + (I + PG)A^{-T}(P - Q) = 0$$

**Lemma 6.** *Suppose $A$ is invertible and $P$ satisfies the Riccati equation*

$$\begin{bmatrix} -P & I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

*then $I + PG$ is invertible.*

**Proof.** Suppose $I + PG$ is not invertible. Then there exists $v \neq 0$ such that $v^T(I + PG) = 0$. Then, using the Riccati equation, we have

$$v^T(-PA + (I + PG)A^{-T}(P - Q)) = 0$$

Then by assumption $v^T(I + PG) = 0$ and so

$$v^T PA = 0$$

and since $A$ is invertible we have $v^T P = 0$. But again, since $v^T(I + PG) = 0$ this implies $v = 0$, which is a contradiction. ∎

We can write the Riccati equation in several different ways

- $-PA + (I + PG)A^{-T}(P - Q) = 0$
- $A^T(I + PG)^{-1}PA - P + Q = 0$
- $A^TPA - P - A^TPG(I + PG)^{-1}PA + Q = 0$
- $A^TPA - P - A^TPB(R + B^TPB)^{-1}B^TPA + Q = 0$

which are easily seen to be equivalent.

**Theorem 7.** *Suppose*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*with $A$ invertible, and $D^TD > 0$ and $C^TD = 0$. Let $Q = C^TC$ and $R = D^TD$ and suppose $P$ satisfies the Riccati equation*

$$A^TPA - P - A^TPB(R + B^TPB)^{-1}B^TPA + Q = 0$$

*and let $D_1D_2 = D^TD$ and*

$$H_1 = \left[\begin{array}{c|c} A^{-T} & -PBD_2^{-1} \\ \hline B^TA^{-T} & D_1 \end{array}\right] \qquad H_2 = \left[\begin{array}{c|c} A & B \\ \hline D_1^{-1}B^TA^{-T}(P - Q) & D_2 \end{array}\right]$$

*Then $\tilde{G}G = H_1H_2$.*

**Proof.** This follows by applying Theorem 4 to the realization of $\tilde{G}G$ given by Lemma 5. ■

**Theorem 8.** *Suppose*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*with $A$ invertible, and $D^TD > 0$ and $C^TD = 0$. Let $Q = C^TC$ and $R = D^TD$, and suppose $P$ satisfies the Riccati equation*

$$A^TPA - P - A^TPB(R + B^TPB)^{-1}B^TPA + Q = 0$$

*Then let $W = R + B^TPB$ and*

$$H_2 = \left[\begin{array}{c|c} A & B \\ \hline W^{-\frac{1}{2}}B^TPA & W^{\frac{1}{2}} \end{array}\right]$$

*Then $\tilde{G}G = \tilde{H}_2H_2$. Furthermore*

$$H_2^{-1} = \left[\begin{array}{c|c} A - BW^{-1}B^TPA & -BW^{-\frac{1}{2}} \\ \hline W^{-1}B^TPA & W^{-\frac{1}{2}} \end{array}\right]$$

**Proof.** This is shown simply by picking $D_1 = RW^{-\frac{1}{2}}$ and $D_2 = W^{\frac{1}{2}}$ and applying Theorem 7. We have

$$\tilde{H} = \left[\begin{array}{c|c} A^{-T} & -PBW^{-\frac{1}{2}} \\ \hline B^T A^{-T} & W^{\frac{1}{2}} - B^T PBW^{-\frac{1}{2}} \end{array}\right]$$

and we need to show that $H = H_2$ and $\tilde{H} = H_1$. To show that $H = H_2$, we only need to check the '$C$' term, which is

$$\begin{aligned} D_1^{-1}(-B^T A^{-T} C^T C + B^T A^{-T} P) &= D_1^{-1} B^T A^{-T} (P - Q) \\ &= D_1^{-1} B^T (I + PG) PA \\ &= D_1^{-1} B^T (I + PBR^{-1}B^T)^{-1} PA \\ &= D_1^{-1} (I + B^T PBR^{-1})^{-1} B^T PA \\ &= D_1^{-1} R(R + B^T PB)^{-1} B^T PA \\ &= D_1^{-1} RW^{-1} B^T PA \\ &= W^{-\frac{1}{2}} B^T PA \end{aligned}$$

as desired. To show that $\tilde{H} = H_1$ we only need to check the '$D$' term, which is

$$\begin{aligned} W^{\frac{1}{2}} - B^T PBW^{-\frac{1}{2}} &= (W - B^T PB)W^{-\frac{1}{2}} \\ &= RW^{-\frac{1}{2}} \\ &= D_1 \end{aligned}$$

as desired. ■

If $P$ satisfies the Riccati equation, then range $\begin{bmatrix} I \\ P \end{bmatrix}$ is an invariant subspace of the Hamiltonian $H$. Defining

$$\hat{H} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

and the Hamiltonian is the generator matrix for $(\tilde{G}G)^{-1}$. So as in the factorization Lemma 1, the inverse of $H_2$ has a generator matrix given by $\hat{H}_{11}$. Specifically, from Theorem 8

$$H_2^{-1} = \left[\begin{array}{c|c} A - BW^{-1}B^T PA & -BW^{-\frac{1}{2}} \\ \hline W^{-1}B^T PA & W^{-\frac{1}{2}} \end{array}\right]$$

Simple algebra, and using the Riccati equation, gives

$$\begin{aligned} \hat{H}_{11} &= A + GA^{-T}(P - Q) \\ &= A - G(I + PG)^{-1} PA \\ &= (I + GP)^{-1} A \end{aligned}$$

and using $G = BR^{-1}B^T$ we have

$$\hat{H}_{11} = A - BW^{-1}B^T PA$$

which, as expected, is the generator matrix of $H_2^{-1}$. Also, because $H$ and $\hat{H}$ are similar, they have the same eigenvalues, and by the invariant subspace property $\hat{H}$ is triangular, so the eigenvalues of $\hat{H}_{11}$ are just a subset of those of $H$. Each solution of $P$ the Riccati equation gives an invariant subspace of $H$, and a corresponding subset of the eigenvalues of $H$ in $\hat{H}_{11} = (I + GP)^{-1}A$.

**Lemma 9.** *Suppose $Q, G$ are symmetric and*

$$H = \begin{bmatrix} A + GA^{-T}Q & -GA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix}$$

*and $H$ has no eigenvalues on the unit circle. Then there is at most one matrix $P$ satisfying the Riccati equation*

$$\begin{bmatrix} -P & I \end{bmatrix} H \begin{bmatrix} I \\ P \end{bmatrix} = 0$$

*such that*

$$\rho\big((I + GP)^{-1}A\big) < 1$$

*This solution is called the* ***stabilizing*** *solution.*

We won't prove this here; for details see Zhou, Doyle and Glover.

**Lemma 10.** *Suppose $Q, G$ are symmetric and*

$$H = \begin{bmatrix} A + GA^{-T}Q & -GA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix}$$

*and $H$ has no eigenvalues on the unit circle. Then the stabilizing solution $P$ of the Riccati equation is symmetric.*

**Proof.** Let $T = \begin{bmatrix} I \\ P \end{bmatrix}$. Then since $P$ satisfies the Riccati equation, we have

$$HT = T\hat{H}_{11} \tag{2}$$

where $\hat{H}_{11} = (I + GP)^{-1}A$. Multiply this equation on the left by $\hat{H}_{11}^* T^* J$ to give

$$\begin{aligned} \hat{H}_{11}^* T^* J T \hat{H}_{11} &= \hat{H}_{11}^* T^* J H T \\ &= T^* H^* J H T \end{aligned}$$

Since $H$ is symplectic, we know $H^T J H = J$. Hence

$$\hat{H}_{11}^* T^* J T \hat{H}_{11} - T^* J T = 0$$

Let $V = T^* J T$. This is a Lyapunov equation in $V$, and since $P$ is by assumption the stabilizing solution, we have $\hat{H}_{11}$ is stable. Hence the unique solution to the Lyapunov equation $\hat{H}_{11}^* V \hat{H}_{11} - V = 0$ is $V = 0$. Hence $T^* J T = 0$, which means $P = P^*$. ■

**Lemma 11.** *Suppose $Q, G$ are symmetric and*

$$H = \begin{bmatrix} A + GA^{-T}Q & -GA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix}$$

*and $H$ has no eigenvalues on the unit circle. Then the stabilizing solution $P$ of the Riccati equation satisfies $P \geq 0$.*

See Zhou, Doyle and Glover for a proof.