4 Fourier Series

4.1 Orthonormal families

Suppose H is a Hilbert space, and we have a sequence of vectors $\phi_0, \phi_1, \dots \in H$ which satisfy

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}$$

then we call the sequence $\phi_0, \phi_1, \dots$ an ***orthonormal family***.

We'd like to write a vector $x \in H$ in terms of the basis H, and to do this we need to know about convergence.

Theorem 1. *Suppose $x_0, x_1, \dots$ is a sequence in $\mathbb{C}$. The series*

$$\sum_{k=0}^{\infty} \phi_k x_k$$

converges if and only if $x \in \ell_2$.

Proof. Define the partial sums

$$s_n = \sum_{k=0}^{n} \phi_n x_n \qquad \text{and} \qquad t_n = \sum_{k=0}^{n} |x_k|^2$$

Notice that $x \in \ell_2$ if and only if the series

$$\sum_{k=0}^{\infty} |x_k|^2$$

converges which holds if and only if the sequence $t_0, t_1, \dots$ converges, which in turn holds if and only if it is Cauchy. Similarly, the series

$$\sum_{k=0}^{\infty} \phi_k x_k$$

converges if and only if the sequence $s_0, s_1, \dots$ converges, which holds if and only if it is Cauchy.

Suppose $m > n$, then we have

$$\begin{aligned} \|s_m - s_n\|_2^2 &= \sum_{k=n+1}^{m} |x_k|^2 \\ &= \|t_m - t_n\|_2^2 \end{aligned}$$

and so the sequence s is Cauchy if and only if t is Cauchy, which holds if and only if $x \in \ell_2$, as desired. ∎

So we can define the linear map

$$U : \ell_2(\mathbb{Z}_+) \to H$$

by

$$Ux = \sum_{k=0}^{\infty} \phi_k x_k$$

Theorem 2. *Some properties of U.*

(i) U is bounded

(ii) The adjoint of U is a map $U^* : H \to \ell_2$, given by

$$(U^*y)_k = \langle \phi_k, y \rangle$$

(iii) $U^*U = I$, that is, U is an isometry.

Proof. To see part (i) we just evaluate the norm

$$\begin{aligned}
\|Ux\|^2 &= \left\| \sum_{k=0}^{\infty} \phi_k x_k \right\|^2 \\
&= \left\| \lim_{n\to\infty} \sum_{k=0}^{n} \phi_k x_k \right\|^2 \\
&= \lim_{n\to\infty} \left\| \sum_{k=0}^{n} \phi_k x_k \right\|^2 \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} |x_k|^2 \\
&= \|x\|^2
\end{aligned}$$

That is, $\|Ux\| = \|x\|$ for all $x \in \ell_2$, and hence $\|U\| = 1$.

For part (ii), we have that by definition the adjoint is the unique linear map satisfying

$$\begin{aligned}
\langle y, Ux \rangle &= \langle y, \sum_{k=0}^{\infty} \phi_k x_k \rangle \\
&= \sum_{k=0}^{\infty} \langle y, \phi_k \rangle x_k
\end{aligned}$$

where we used continuity of the inner-product. Hence

$$\langle y, Ux \rangle = \langle z, x \rangle$$

where $z \in \ell_2$ is

$$z_k = \langle \phi_k, y \rangle$$

To see part (iii), we have

$$\begin{aligned}
(U^*Ux)_i &= \langle \phi_i, \sum_{j=1}^{\infty} \phi_j x_j \rangle \\
&= \sum_{j=1}^{\infty} \langle \phi_i, \phi_j \rangle x_j
\end{aligned}$$

using continuity of the inner-product. Hence

$$(U^*Ux)_i = x_i$$

as desired. ∎

An orthonormal family is called an ***orthonormal basis*** for H if for every $f \in H$ there exists a sequence $x \in \ell_2$ such that

$$f = \sum_{k=0}^{\infty} \phi_k x_k$$

Theorem 3. *The orthonormal family $\phi_0, \phi_1, \dots$ is an orthnormal basis if and only if $UU^* = I$.*

Proof. First we'll show the *if* direction. Suppose $UU^* = I$. Then

$$f = UU^*f = \sum_{k=0}^{\infty} \phi_k (U^*f)_k$$

which gives f as a linear combination of the basis functions. To show the converse, suppose there exists the desired $x_0, x_1, \dots$ so that f is

$$f = \sum_{k=0}^{\infty} \phi_x x_k$$

This means $f = Ux$ and therefore $U^*f = U^*Ux = x$. Hence $f = UU^*f$. Since $\phi_0, \phi_1, \dots$ is a basis such an expansion exists for every $f \in H$, hence $UU^* = I$. ∎

4.2 Fourier Series

Consider the space $L_2(\mathbb{T})$ of functions mapping $\mathbb{T}$ to $\mathbb{C}$ with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(e^{j\theta})} g(e^{j\theta})\, d\theta$$

Theorem 4. *The sequence of functions $\dots, \phi_{-1}, \phi_0, \phi_1, \dots$ where $\phi_k : \mathbb{T} \to \mathbb{C}$ is given by*

$$\phi_k(e^{j\theta}) = e^{jk\theta}$$

is an orthonormal basis for $L_2(\mathbb{T})$.

Proof. This is not too hard a proof, but would take us too far from our desired course, so we'll omit it. Orthonormality is easy; all we would need to show is completeness. See e.g. p. 45 of Young, or p. 61 of Luenberger. ∎

This result dates back to 1907, with the work of Riesz and Fischer. You have probably seen results on convergence of Fourier series before. The simplest results state that the partial Fourier sums of f converge pointwise if f is twice continuously differentiable. The famous result of Carleson in 1966 is that the Fourier series of any L_2 function converges pointwise almost everywhere.

Notice that we can also write

$$\phi_k(\lambda) = \lambda^k$$

for $\lambda \in \mathbb{T}$. We can also write the inner product as the complex integral

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} \overline{f(\lambda)} g(\lambda) \lambda^{-1} \, d\lambda$$

We'll define the map $F : \ell_2(\mathbb{Z}) \to L_2(\mathbb{T})$ to be

$$Fx = \sum_{k=-\infty}^{\infty} x_k \phi_k$$

Then F^* takes a function on $\mathbb{T}$ and returns the coefficients of its ***Fourier series***. The map F, which is the inverse Fourier transform, is sometimes called the ***λ-transform***.

4.3 Examples and rational functions

Define for convenience the ***open unit disk***

$$\mathbb{D} = \{ \lambda \in \mathbb{C} \mid |\lambda| < 1 \}$$

and its closure

$$\bar{\mathbb{D}} = \{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \}$$

Some examples. If $|a| < 1$ then

$$x = (\dots, 0, 0, 0, \boxed{1}, a, a^2, a^3, \dots) \qquad \Longrightarrow \qquad (Fx)(\lambda) = \frac{1}{1 - a\lambda}$$

On the other hand, if $|a| > 1$ then

$$x = (\dots, -a^{-3}, -a^{-2}, -a^{-1}, \boxed{0}, 0, 0, \dots) \qquad \Longrightarrow \qquad (Fx)(\lambda) = \frac{1}{1 - a\lambda}$$

This gives us the Fourier series for all rational functions without poles on $\mathbb{T}$, by partial fractions. If f is rational but has a pole on $\mathbb{T}$, then it is not square-integrable, that is, it is not an element of $L_2(\mathbb{T})$, and so it has no Fourier series expansion. Notice also that if f is rational and has no poles in $\bar{\mathbb{D}}$, then its Fourier series x is zero on negative time, and is a stable exponential on positive time. On the other hand, if f has no poles outside $\mathbb{D}$ then its Fourier series x is zero on positive time, and is a stable exponential in negative time. **There are no unstable signals in ℓ_2, and we cannot use the Fourier theory to represent any unstable signals.**

This is a general feature of Fourier series, and there is an exactly parallel phenomena in continuous time. By working with square-integrable signals, we lose the ability to represent unstable signals. However, we gain the ability to represent functions which are nonnegative on both positive and negative time, which is highly useful for signal processing, where we'd like to use impulses responses which are not necessarily causal. And we gain the ability to formulate and solve optimal control problems, which are infinite dimensional least-squares problems. We'll deal with unstable control systems by stabilizing them first, and then analyzing their norm.

If f is a rational function, we call it ***stable*** if all of its poles are ***outside*** the closed unit disk. You may have seen the opposite definition when using the z***-transform***. If $x : \mathbb{Z}_+ \to \mathbb{C}$, it's z-transform is

$$g(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

For $x \in \ell_2$ with $f = Fx$ we clearly have $f(\lambda) = g(\lambda^{-1})$, hence if p is a pole of the z-transform then p^{-1} is a pole of the Fourier transform f.

4.4 The Hardy Space H_2

So far we have seen that there is a one-to-one isometric map between sequences $x \in \ell_2(\mathbb{Z})$ and functions $f \in L_2(\mathbb{T})$ given by $f = Ux$, which is just

$$f = \sum_{k=-\infty}^{\infty} x_k \phi_k$$

which converges in the L_2 norm on the complex unit circle. Even though $\phi_k(\lambda) = \lambda^k$ for $\lambda \in \mathbb{T}$, in general it doesn't make sense to write the sum as

$$\sum_{k=-\infty}^{\infty} x_k \lambda^k$$

since the sum on the right-hand side may not converge for some $\lambda \in \mathbb{T}$.

But there are times when we can say something about the convergence of this power series, for some λ. One such time is when $x_k = 0$ for all $k < 0$. In other words, we consider those sequences which are zero for negative time, which we may as well view as functions in $\ell_2(\mathbb{Z}_+)$. Since $\ell_2(\mathbb{Z}+)$ is a subspace of $\ell_2(\mathbb{Z})$, the image of $\ell_2(\mathbb{Z}_+)$ under F is therefore a subspace of $L_2(\mathbb{T})$. It is called $\tilde{H}_2$, and F gives a bijection between $\ell_2(\mathbb{Z})$ and $\tilde{H}_2$.

In fact if $x \in \ell_2(\mathbb{Z}_+)$ (notice the + subscript) then the power series also converges absolutely inside the complex unit disk, as follows.

Theorem 5. *Suppose $x \in \ell_2(\mathbb{Z}_+)$. Then the power series*

$$\sum_{k=0}^{\infty} x_k \lambda^k$$

converges absolutely if $|\lambda| < 1$.

Proof. Since $x \in \ell_2$, there exist M such that for all $k \geq 0$ the inequality $|x_k| < M$ holds. Let s_n be the partial sum

$$s_n = \sum_{k=0}^{n} |x_k \lambda^k|$$

Then we have

$$
\begin{aligned}
s_n &\le M \sum_{k=0}^{n} |\lambda^k| \\
&\le M \sum_{k=0}^{\infty} |\lambda^k|
\end{aligned}
$$

Hence the sequence $s_0, s_1, \ldots$ is increasing and bounded, and so convergent. ■

This leads us to define the ***Hardy space*** H_2 as follows.

$$
H_2 = \left\{ f : \mathbb{D} \to \mathbb{C} \mid \text{there exists } x \in \ell_2(\mathbb{Z}) \text{ such that } f(\lambda) = \sum_{k=0}^{\infty} x_k \lambda^k \right\}
$$

From complex analysis, we know that every function $f \in H_2$ is analytic on $\mathbb{D}$, since it has a power series which is absolutely convergent on $\mathbb{D}$ from Theorem 5, and the power series is unique. So we also have a bijection between $\ell_2(\mathbb{Z}_+)$ and H_2.

So we can start with a function $f \in H_2$, can construct it's power series $x \in \ell_2(\mathbb{Z}_+)$, and then construct $g \in \tilde{H}_2$ by $g = Fx$. Or we can start with $g \in \tilde{H}_2$, construct it's Fourier series $x \in \ell_2(\mathbb{Z}_+)$, and then let f be the corresponding analytic function in H_2. So we have a bijection between H_2 and $\tilde{H}_2$. One might guess that f and g are related, and in fact it can be proved that

$$
\lim_{r \to 1^-} f(re^{j\theta}) = g(e^{j\theta}) \qquad \text{for almost all } \theta \in [0, 2\pi]
$$

To summarize, every square-summable discrete-time signal x corresponds to a unique square-integrable function g on the unit circle. But if in addition the signal x is zero on negative time then there is also a corresponding unique *analytic* function f on the open unit disk, and f and g meet up nicely on the circle. In particular, f is analytic on $\mathbb{D}$ implies that it has no poles on $\mathbb{D}$. If f and g are rational, then they will be the ***same*** rational.