2 The ℓ_∞ induced norm

We consider a linear system defined by convolution, where the input u is mapped to the output y by

$$y(t) = \sum_{k=0}^{t} g(t-k)u(k)$$

For any functions $g, u \in \ell_e(\mathbb{Z}, \mathbb{R})$ this makes sense, since the above sum is finite. This therefore gives us a linear map $G : \ell_e \to \ell_e$.

Suppose we know that $u \in \ell_\infty$, that is that the input is *bounded.* For which impulse responses g is the output y also bounded? The following result answers this question.

Theorem 1. *Suppose $g \in \ell_1$. Then the convolution map G defined by*

$$y = Gu \qquad \text{if} \qquad y(t) = \sum_{k=0}^{t} g(t-k)u(k)$$

maps ℓ_∞ to ℓ_∞, and is bounded. Further the induced norm of G is

$$\|G\|_{i\infty} = \|g\|_1$$

Proof. We first show boundedness. Suppose $u \in \ell_\infty$ and $g \in \ell_1$. Then we have

$$\begin{aligned} |y(t)| &= \left| \sum_{k=0}^{t} g(t-k)u(k) \right| \\ &\leq \sum_{k=0}^{t} |g(t-k)||u(k)| \\ &\leq \|u\|_\infty \sum_{k=0}^{t} |g(t-k)| \\ &\leq \|u\|_\infty \|g\|_1 \end{aligned}$$

and since this holds for all t we have

$$\|y\|_\infty \leq \|g\|_1 \|u\|_\infty$$

and therefore $\|G\| \leq \|g\|_1$. We need to show that these norms are equal, and to do this we show that, given $\varepsilon > 0$, there exists $u \in \ell_\infty$ with $\|u\|_\infty \leq 1$ and

$$\|Gu\|_\infty \geq \|g\|_1 - \varepsilon$$

To do this, pick N sufficiently large that

$$\sum_{k=0}^{N} |g(t)| \geq \|g\|_1 - \varepsilon$$

and let

$$u(t) = \begin{cases} \operatorname{sign} g(N-t) & \text{if } 0 \le t \le N \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\begin{aligned} y(N) &= \sum_{k=0}^{N} g(N-k)u(k) \\ &= \sum_{k=0}^{N} |g(N-k)| \\ &\ge \|g\|_1 - \varepsilon \end{aligned}$$

as desired. ∎

Inputs that are arbitrarily close to worst case are also visible from this proof; we simply pick a large N, then flip the impulse response in time, and then takes it sign.

The step response $s \in \ell_e$ corresponding to impulse response g is just the sum

$$s(t) = \sum_{k=0}^{t} g(t)$$

If the impulse response is nonnegative, i.e., $g(t) \ge 0$ for all t, then the worst case input u is just a constant. The impulse response is nonnegative if and only if the step response is non-decreasing.

If, however, the impulse response changes sign, then the worst case input oscillates between $+1$ and -1. Also small high-frequency oscillations in the step response can correspond to large oscillations in the impulse response, and a correspondingly large induced ∞-norm of the convolution map.

It is tempting to look at the frequency response of G, that is the Fourier transform of g, and find the frequency ω_0 at which this has largest magnitude. This gives the largest possible amplification of any sinusoid $q = |\hat{h}(e^{j\omega_0})|$. We call this quantity the ∞-norm of the transfer function

$$\|\hat{g}\|_\infty = \sup_{\theta \in [0, 2\pi)} |\hat{g}(e^{j\theta})|$$

This quantity may be much smaller than the worst possible peak gain. In fact, for any stable rational transfer function

$$\|\hat{g}\|_\infty \le \|g\|_1 \le (2n+1)\|\hat{g}\|_\infty$$

where n is the number of poles of $\hat{g}$. We'll see more on transfer functions later.