9 State-Space Realization

Given a transfer function $G \in \mathbb{R}[z]^{m \times p}$, such as

$$G(z) = \begin{bmatrix} \dfrac{z-1}{z-2} & \dfrac{z-2}{z-3} \\ \dfrac{z}{z^2+1} & \dfrac{z-3}{(z-4)(z-2)} \end{bmatrix}$$

we would like to find matrices A, B, C, D such that

$$G(z) = C(zI - A)^{-1}B + D$$

Such matrices are called a ***realization*** for G. If the matrix A is $n \times n$ then the realization is said to have ***degree*** or ***order*** n. We would like to address a number of questions, such as when does a realization exist, and what is the smallest degree possible for a realization.

9.1 Invariant Subspaces

Suppose $A \in \mathbb{R}^{n \times n}$ and $V \subset \mathbb{R}^n$ is a subspace. The subspace V is called ***invariant*** under A if $AV \subset V$. This means

$$Ax \in V \qquad \text{for all } x \in V$$

We also say V is A-invariant. Some examples are as follows.

- $\{0\}$ and $\mathbb{R}^n$ are invariant under every A.
- If $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $V = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \;\middle|\; v \in \mathbb{R}^r \right\}$ then V is A-invariant.

We'll have many needs for invariant subspaces. For realization theory, we need the following.

Lemma 1. *The controllable subspace*

$$V = \text{range} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

is A-invariant.

Proof. If $x \in V$ then there exist $u_0, \dots, u_{n-1}$ such that

$$x = Bu_0 + ABu_1 + \cdots + A^{n-1}Bu_{n-1}$$

and hence

$$Ax = ABu_0 + ABu_1 + \cdots + A^nBu_{n-1}$$

Now by the Cayley-Hamilton theorem, A^n is a linear combination of $I, A, \dots, A^{n-1}$ and so Ax is also an element of V. ∎

This is intuitively reasonable, since the controllable subspace contains all states which may be reached starting from state zero. If by applying A we could leave this set, then we could reach additional points in the state space.

Returning to the general theory of invariant subspaces, we have the following result.

Lemma 2. *Suppose $A \in \mathbb{R}^{n\times n}$, and $M \in \mathbb{R}^{n\times k}$. Then* range M *is A-invariant if and only if there exists $X \in \mathbb{R}^{k\times k}$ such that*

$$AM = MX$$

Proof. To see *if*, suppose $x \in \operatorname{range} M$. Then there exists z such that $x = Mz$, and so $Ax = AMz$. We know that there exists X such that $AM = MX$, and so $Ax = MXz$ hence $Ax \in \operatorname{range} M$.

For the *only if* direction, we construct X as follows. Suppose

$$M = \begin{bmatrix} q_1 & \dots & q_k \end{bmatrix}$$

Then $q_1 \in \operatorname{range} M$ and since $\operatorname{range} M$ is A-invariant we have $Aq_1 \in \operatorname{range} M$. Therefore there exists w_1 such that $Aq_1 = Mw_1$. Repeating this for each of the q_i gives

$$AM = M \begin{bmatrix} w_1 & \dots w_k \end{bmatrix}$$

and so we let $X = \begin{bmatrix} w_1 & \dots w_k \end{bmatrix}$. ∎

This result allows us to show that an invariant subspace gives coordinates T with respect to which the linear map A is upper triangular, as follows.

Lemma 3. *Suppose $A \in \mathbb{R}^{n\times n}$ and V is a k-dimensional invariant subspace. Let $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ be invertible with* range $T_1 = V$. *Then there exists $X \in \mathbb{R}^{k\times k}$ and Y, Z such that*

$$T^{-1}AT = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

Proof. Suppose V is A-invariant, and let $T \in \mathbb{R}^{n\times n}$ be partitioned as

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$$

where the columns of T_1 are a basis for V, and the columns of T_2 complete the basis to span $\mathbb{R}^n$, so that T is invertible. Then $\operatorname{range} T_1$ is A-invariant, hence by the previous lemma there exists X such that

$$AT_1 = T_1 X$$

Therefore

$$AT = T \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

∎

We saw the earlier example that block triangular matrices have a natural invariant subspace, and this result now shows that all invariant subspaces have this form, in the appropriate coordinates. Similarly, by swapping T_1 and T_2, we can use lower triangular matrices instead. This also means that if A is *lower triangular*, not just block triangular, then for every r we have

$$\operatorname{range} \begin{bmatrix} 0 \\ I_r \end{bmatrix} \text{ is } A\text{-invariant}$$

This is causality. Notice also that if

$$T^{-1}AT = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

then

$$\det(\lambda I - A) = \det(\lambda I - X)\det(\lambda I - Z)$$

and so the eigenvalues of X are a subset of those of A.

9.2 Minimal Realizations

Lemma 4. *Suppose* $T_1 \in \mathbb{R}^{n\times k}$ *and*

$$\operatorname{range} T_1 = \operatorname{range}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \qquad \operatorname{null} T_1 = \{0\}$$

and complete the basis with a matrix T_2 *so that* $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ *is invertible. Then*

$$T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \qquad T^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

Proof. The triangular form of $T^{-1}AT$ follows from Lemma 1. Suppose B has columns

$$B = \begin{bmatrix} b_1 & \dots & b_r \end{bmatrix}$$

Since $\operatorname{range} B \subset \operatorname{range} T_1$ there exists x_i such that

$$b_i = T_1 x_i$$

and hence

$$b_i = T \begin{bmatrix} x_i \\ 0 \end{bmatrix}$$

Let $\hat{B}_1 = \begin{bmatrix} x_1 & \dots & x_r \end{bmatrix}$, then $B = T \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$ as desired. ∎

Suppose

$$\hat{G}(z) = C(zI - A)^{-1}B + D$$

We define the notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \hat{G}$$

This block-matrix notation means the *rational function*, and we can also write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = C(zI - A)^{-1}B + D$$

Given a state-space realization, we may change coordinates as follows. It is easy to see that if T is invertible, then

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$$

The interpretation of this result is that if y and u are related by state-space dynamics

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned}$$

and we change coordinates according to $x_t = Tz_t$, then

$$\begin{aligned} z_{t+1} &= T^{-1}ATz_t + T^{-1}Bu_t \\ y_t &= CTz_t + Du_t \end{aligned}$$

so with initial conditions $x_0 = 0$ and $z_0 = 0$ both systems map inputs to outputs in the same way. As well as the same transfer function, these systems have the same impulse response

$$H_t = \begin{cases} D & \text{if } t = 0 \\ CA^{t-1}B & \text{if } t \geq 0 \end{cases}$$

We call two realizations A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 *equivalent* if they have the same transfer function

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

This leads to following result.

Lemma 5. *If A, B is not controllable, then there exists a realization with smaller state dimension.*

Proof. If A, B is not controllable, then in the coordinates defined by Lemma 4 we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and hence

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

■

The following results holds for observability, in exactly the same way. If A, C is not controllable, let V be the unobservable space

$$V = \operatorname{null} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

then V is A-invariant, and so there exist coordinates in which

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

in which case

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

So far, we know that

$$A, B, C \text{ is minimal} \qquad \Longrightarrow \qquad \begin{array}{l} (A,B) \text{ is controllable, and} \\ (A,C) \text{ is observable} \end{array}$$

In fact, the converse is true, as we shall now show. First, we need a preliminary result.

Lemma 6. *Suppose $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. Then*

$$\operatorname{rank} AB = \operatorname{rank} B - \dim(\operatorname{null} A \cap \operatorname{range} B)$$

Proof. Let $X = \begin{bmatrix} x_1 & \dots & x_s \end{bmatrix}$ be a basis for $A \cap \operatorname{range} B$. Complete this basis with $Z = \begin{bmatrix} z_1 & \dots & z_t \end{bmatrix}$ so that $\begin{bmatrix} X & Z \end{bmatrix}$ is a basis for $\operatorname{range} B$.

Then we have $\operatorname{range} AB = \operatorname{range} A \begin{bmatrix} X & Z \end{bmatrix} = \operatorname{range} AZ$. Also $\operatorname{null} AZ = \{0\}$, since if not then we would have $q \in \operatorname{null} AZ$, then $Zq \in \operatorname{null} A$, and this is not true by our construction of Z. Hence $\dim \operatorname{range} AB = t$, as desired. ■

Corollary 7. *Suppose $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. Then*

(i) $\operatorname{rank} AB \leq \operatorname{rank} B$

(ii) $\operatorname{rank} AB \leq \operatorname{rank} A$

(iii) $\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - n$

Proof. Part (i) is immediate, and part (ii) follows by taking the transpose. For part (iii) we have

$$\begin{aligned} \dim(\operatorname{null} A \cap \operatorname{range} B) &\leq \dim \operatorname{null} A \\ &= n - \dim \operatorname{range} A \\ &= n - \operatorname{rank} A \end{aligned}$$

since $\dim \operatorname{range} A + \dim \operatorname{null} A = n$. ■

This gives us the main result.

Theorem 8. *Suppose $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{n \times p}$. Suppose $k \geq n-1$ and where*

$$\Gamma = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^k B \end{bmatrix}$$

Then

(i) If A, B, C is controllable and observable then

$$n \leq \operatorname{rank} \Gamma$$

(ii) For any A, B, C we have

$$\operatorname{rank} \Gamma \leq n$$

Proof. Define for convenience

$$P = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} \qquad Q = \begin{bmatrix} B & AB & \dots & A^k B \end{bmatrix}$$

For part (i), controllability implies $\operatorname{rank} Q = n$ and observability implies $\operatorname{rank} P = n$, and so from Corollary 7 we have

$$\begin{aligned} \operatorname{rank} PQ &\geq \operatorname{rank} P + \operatorname{rank} Q - n \\ &\geq n \end{aligned}$$

which is the desired result. For part (ii) we have $\operatorname{rank} PQ \leq \operatorname{rank} P$, and P has n columns, so $\operatorname{rank} P \leq n$. ∎

The matrix Γ is called the ***Hankel operator*** corresponding to the system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

A matrix is called *Hankel* if it is constant along anti-diagonals. This result has several important consequences. First, we have the following.

Corollary 9. *A, B, C, D is minimal if and only if A, B is controllable and A, C is observable.*

Proof. We only need to show the *if* direction. Suppose A, B is controllable and A, C is observable. Then Theorem 8 implies that $\operatorname{rank} \Gamma = n$. Also part (ii) of the theorem implies that every realization for this system has a number of states greater than $\operatorname{rank} \Gamma$. ∎

The entries of the Hankel operator are just the impulse response coefficients of a system. Theorem 8 then implies that the rank of the Hankel operator is the number of states of a minimum realization. Notice also that the entries of the Hankel operator are independent of the particular realization.

The Hankel operator may be interpreted as follows. The current state is related to past inputs via the controllability matrix

$$x(0) = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} \begin{bmatrix} u(-1) \\ u(-2) \\ u(-3) \\ \vdots \end{bmatrix} = \mathcal{C} u_{\text{past}}$$

Suppose that at $t = 0$ we turn off the input, then future measurements are related to the initial state via the observability matrix according to

$$y_{\text{future}} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} x(0) = \mathcal{O}x(0)$$

Therefore, the Hankel operator relates the past inputs to the future outputs by

$$y_{\text{future}} = \mathcal{O}\mathcal{C}u_{\text{past}} = \Gamma u_{\text{past}}$$

If $\rho(A) < 1$ then the Hankel operator is bounded operator from $\ell_2(\mathbb{Z}_-)$ to $\ell_2(\mathbb{Z}_+)$. If we partition the corresponding bi-infinite Toeplitz matrix L_g according to

$$L_g = \begin{bmatrix} S_g^{\text{flip}} & 0 \\ \Gamma^{\text{flip}} & S_g \end{bmatrix}$$

where S_g is the semi-infinite Toeplitz operator, and the superscript *flip* indicates that we reverse the order of inputs and/or outputs.

Then the rank of Γ tells us the smallest inside dimension of a factorization of Γ, which we can interpret as the amount of memory required about the past inputs necessary to generate the future outputs.

There is an extensive theory of Hankel operators. Note that the Hankel operator may be obtained directly from data; then we can use the Hankel matrix to compute a realization for the system, called *subspace identification*. One may also approximate Γ to generate approximate realizations for a system, via the theory of *model reduction*. There is also an analogue for nonlinear and finite state systems, called *Nerode realization theory*.

9.3 Realization Algorithms

Given a rational transfer matrix G we would like to find A, B, C, D such that

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

For a scalar transfer function, there is a simple formula, as follows.

$$g(z) = \frac{c_{n-1}z^{n-1} + \cdots + c_1 z + c_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

Then $g(z) = C(zI - A)^{-1}B$ if

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix}$$

If the numerator and denominator polynomials have no common roots then this realization is minimal. Also, if the numerator polynomial has the same degree as the denominator, one can use the division algorithm to compute the quotient and remainder, and set D to the remainder.

For matrix transfer functions, we construct realizations an entry at a time as follows. Suppose

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

Then

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right] \qquad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right]$$

A procedure for realization of a rational transfer matrix $\hat{G}$ is

1. Realize each element $\hat{G}_{ij}$, which is a scalar transfer function.
2. Realize the columns.
3. Realize the row of columns.

The resulting realization may be non-minimal. For example,

$$\hat{G}(z) = \begin{bmatrix} \dfrac{1}{z+1} & \dfrac{2}{z+1} \end{bmatrix}$$

The previous construction leads to

$$\hat{G}(z) = \left[\begin{array}{rr|rr} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 1 & 2 & 0 & 0 \end{array}\right]$$

but a lower-order realization is

$$\hat{G}(z) = \left[\begin{array}{r|rr} -1 & 1 & 2 \\ \hline 1 & 0 & 0 \end{array}\right]$$

9.4 Calculus of Rationals

We can compute realizations for products, sums and inverses as follows. Two realizations for the product are

$$\begin{aligned} G_1 G_2 &= \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right] \end{aligned}$$

To prove this, use the fact that

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

For the sum we have

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

We can also invert a realization. If D is invertible, then

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

Note that this is a statement about *rational* functions; specifically it means that

$$\left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = I$$

where we multiply the two rational functions and cancel all common factors. Similarly, if D is left invertible, then

$$\left[\begin{array}{c|c} A - BD^{\dagger}C & -BD^{\dagger} \\ \hline D^{\dagger}C & D^{\dagger} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = I$$