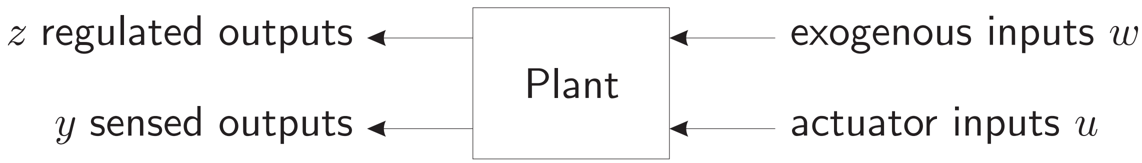
Engr210a Lecture 12: LFTs and stability

- 2-input 2-output framework
- Example problem formulations
- Linear fractional transformations
- Well-posedness
- Realizability
- Internal stability
- Input-output characterization of internal stability

2-input 2-output framework



Inputs

- Actuator inputs u are those inputs to the system that can be manipulated by the controller.
- Exogenous inputs w are all other inputs.

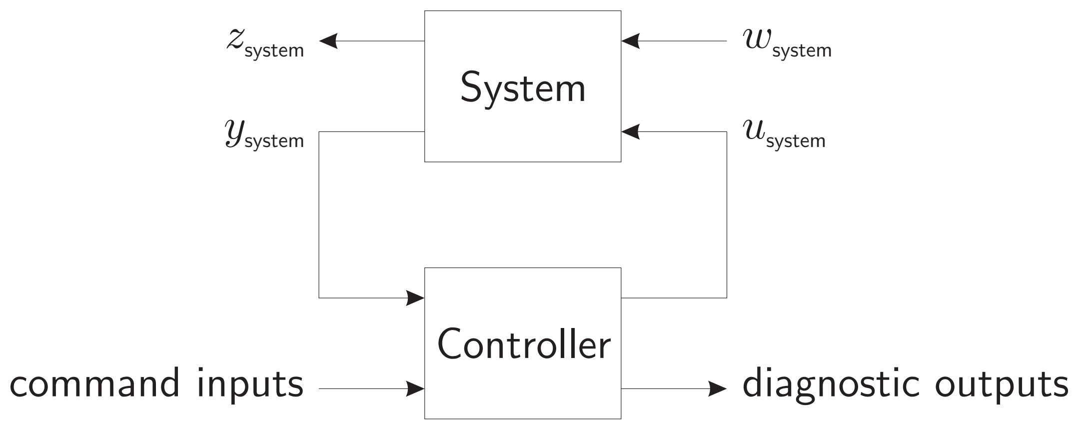
Outputs

- Regulated outputs z are every output signal from the model.
- Sensed outputs are those outputs which are accessible to the controller.

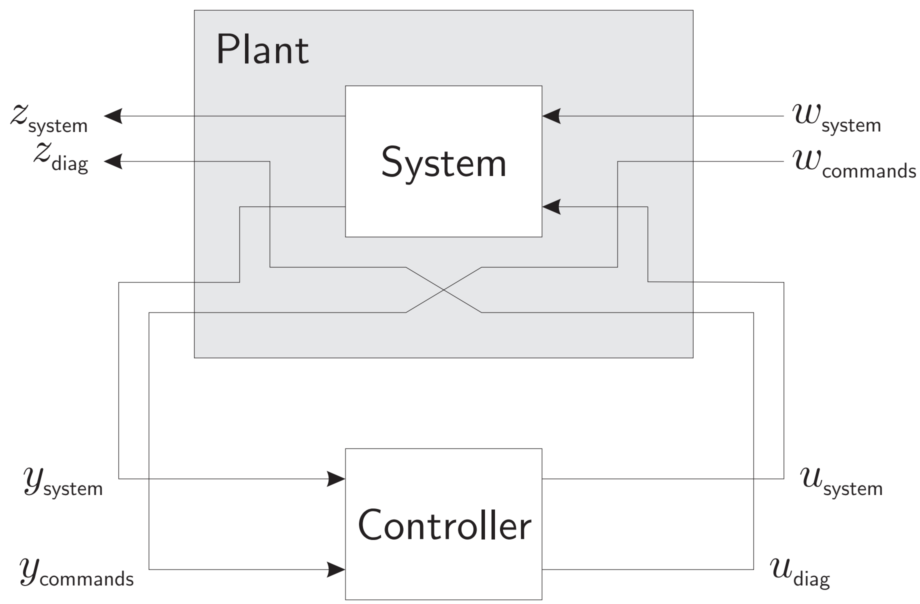
Notes

- Objective is to write all specifications in terms of z and w.

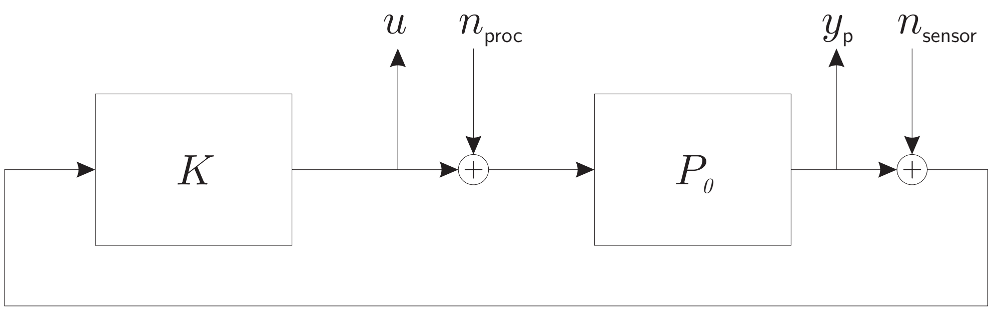
Command inputs and diagnostic outputs



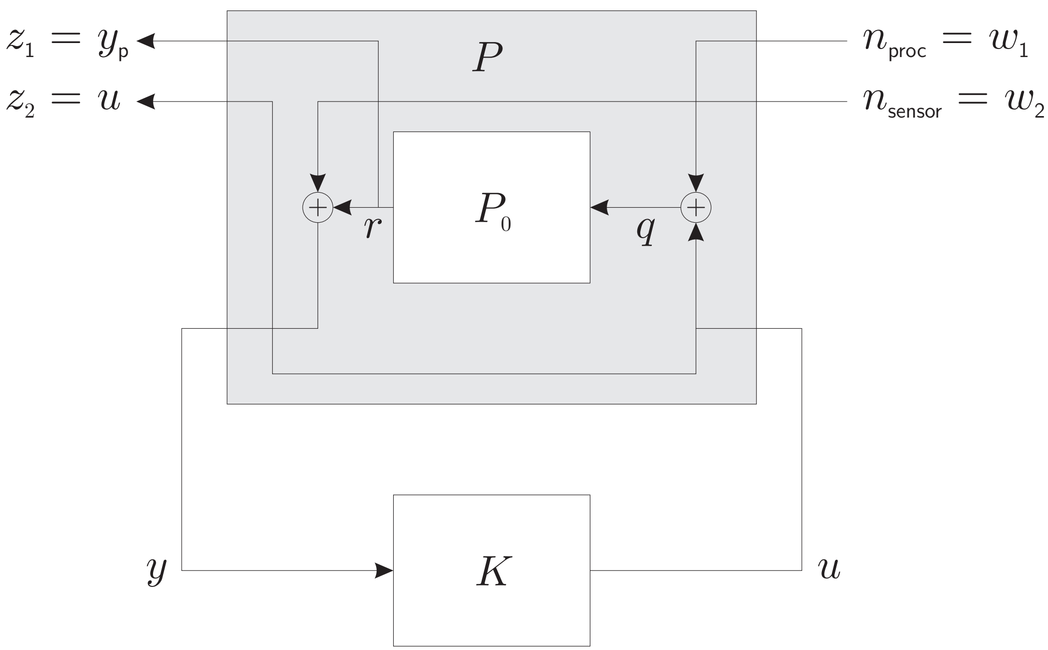
Formulate the above as



Example: the regulator



Formulate the above as



The plant P is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} P_0 & 0 & P_0 \\ 0 & 0 & 1 \\ P_0 & 1 & P_0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$$

Suppose P_0 is

$$\dot{x} = Ax + Bq$$
$$r = Cx + Dq$$

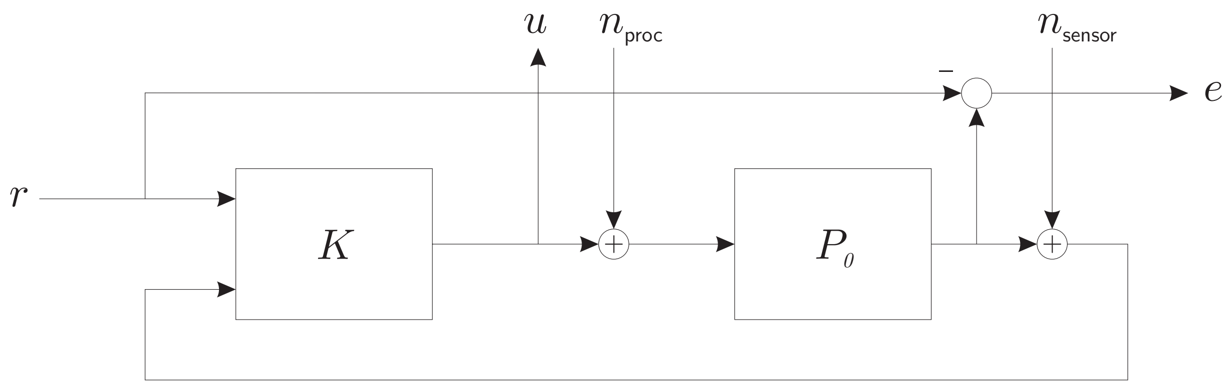
Substituting

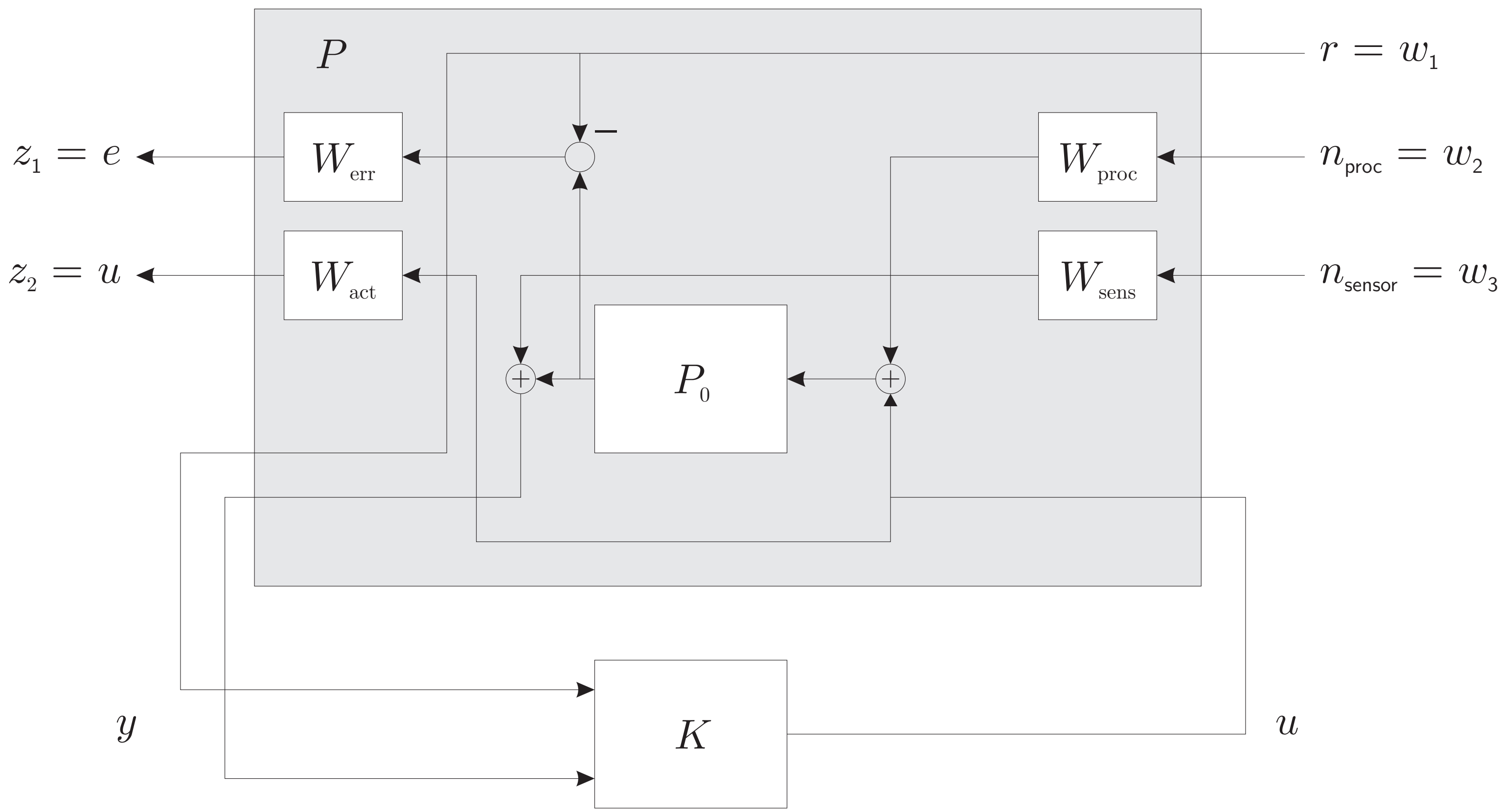
$$z_2 = u \qquad q = w_1 + w_2$$
$$z_1 = r \qquad y = r + w_2$$

leads to

$$P = \left[\begin{array}{c|ccc} A & B & 0 & B \\ \hline C & D & 0 & D \\ 0 & 0 & 0 & I \\ C & D & I & D \end{array}\right]$$

Example: a tracking problem





Linear fractional transformations

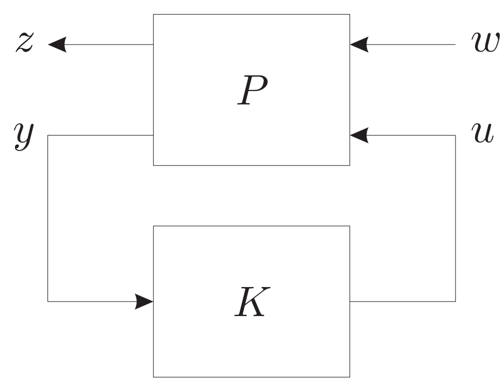
Suppose P and K are state-space systems with

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad \hat{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

and

$$u = Ky \qquad \text{where} \qquad \hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

The following interconnection is called the *(lower) star-product* of P and K, or the *(lower) linear-fractional transformation* (LFT).

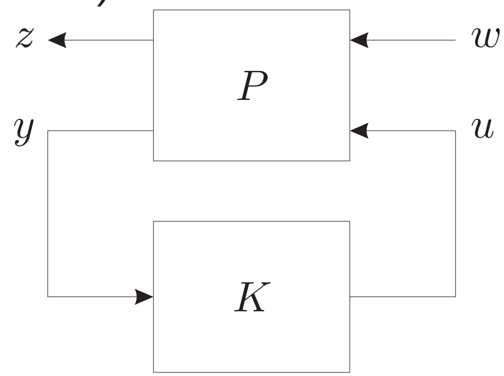


The map from w to z is given by

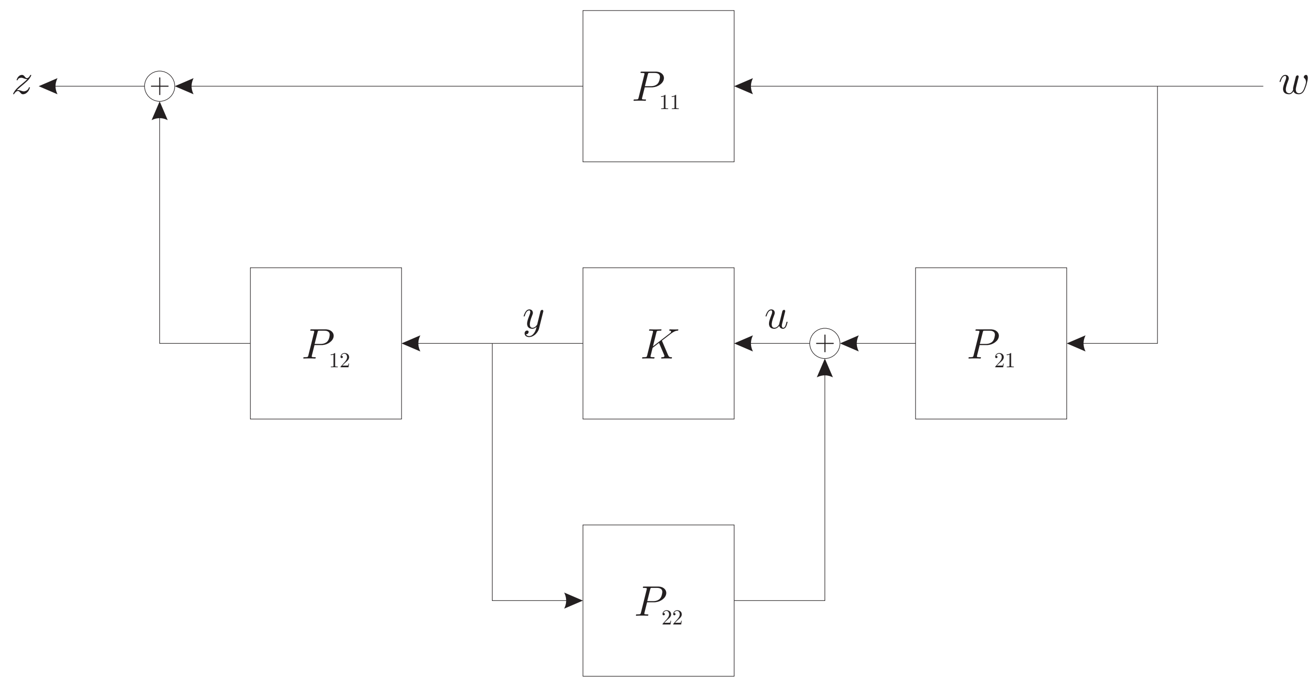
$$\underline{S}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Linear fractional transformations

The following interconnection is called the *(lower) star-product* of P and K, or the *(lower) linear-fractional transformation* (LFT).



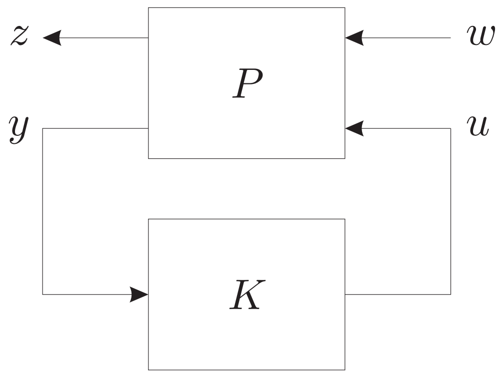
Equivalently



Linear fractional transformations

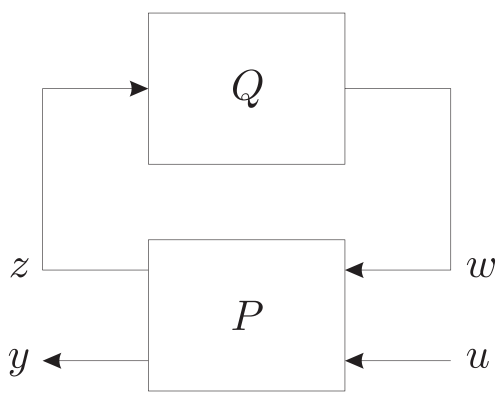
Lower LFT

$$\underline{S}(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

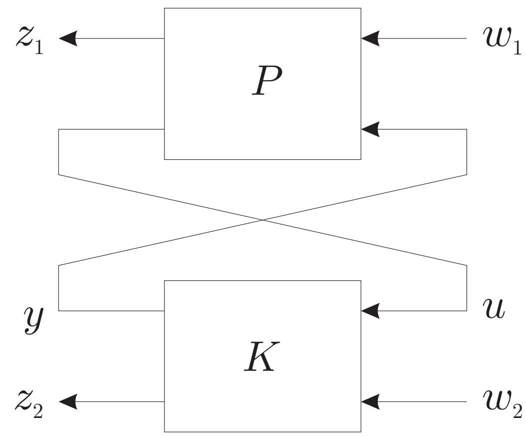


Upper LFT

$$\overline{S}(P,K) = P_{22} + P_{21}Q(I - P_{11}Q)^{-1}P_{12}$$



Star Product



$$S(P,K) = \begin{bmatrix} \underline{S}(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \overline{S}(K, P_{22}) \end{bmatrix}$$

General problem

A very general and useful way to formulate control problems is the following.

$$\begin{array}{rl} \text{minimize} & \|\underline{S}(P,K)\| \\ \text{subject to} & \text{The closed-loop is stable} \end{array}$$

Notes

- Many different norms can be used; the two most common are the H_2 and H_∞ norm.
- Robustness specifications can also be put into this form; we will see much more later.

Well-posedness

The state-space equations are

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{aligned} \qquad \begin{aligned} \dot{x}_K(t) &= A_K x(t) + B_K y(t) \\ u(t) &= C_K x(t) + D_K y(t) \end{aligned}$$

For linear operators or transfer functions, we have

$$\underline{S}(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

Caveat: invertibility here is as a transfer matrix, not as a bounded operator.

In state-space this is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x_K}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t)$$

$$z(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + D_{11} w(t)$$

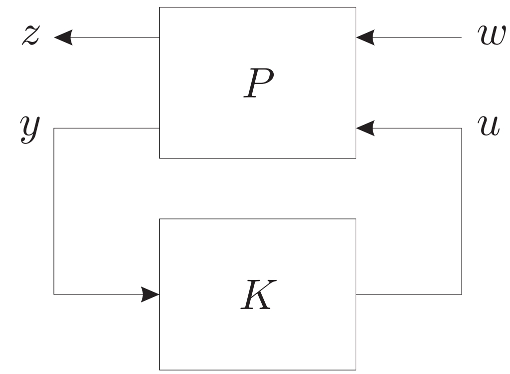
where

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$

We need to solve these equations for $u(t)$ and $y(t)$.

Well-posedness

The LFT interconnection is called *well-posed* if unique solutions exist for u and y, given



- any initial conditions $x(0)$ and $x_K(0)$,
- any smooth input function w,
- any small perturbations to the state-space matrices for P and K.

Theorem

The LFT interconnection is well-posed $\iff$ $I - D_{22}D_K$ is invertible.

Notes

- This follows from $$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$
- Note that $\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix}$ where $Q = (I - D_{22}D_K)^{-1}$.
- In frequency domain: $\lim_{w\to\infty} \hat{P}_{22}(j\omega) = D_{22}$ and $\lim_{w\to\infty} \hat{K}(j\omega) = D_K$

Well-posedness

- The interconnection is well-posed iff $I - D_{22}D_K$ is invertible.
- The interconnection is well-posed iff $I - D_K D_{22}$ is invertible.
- In frequency domain, we have $\lim_{w \to \infty} \hat{P}_{22}(j\omega) = D_{22}$ and $\lim_{w \to \infty} \hat{K}(j\omega) = D_K$.
- If P and K are rational, then the LFT interconnection is well-posed if and only if there exists $w \in \mathbb{R}$ such that

$$\det\big(I - \hat{P}_{22}(j\omega)\hat{K}(j\omega)\big) \neq 0$$

- If $D_K = 0$, that is if $\hat{K}$ is strictly proper, then the system is well-posed.
- If $D_{22} = 0$, that is if $\hat{P}_{22}$ is strictly proper, then the system is well-posed.

Notes

- We require well-posedness so that the system equations make sense.
- Physical systems are always well-posed; roughly, if $\hat{P}$ is a physical system then $\hat{P}$ must be strictly proper.
- Well-posedness says nothing about stability.

Realizability

A general problem can be written as

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP \\ & \text{The closed-loop is stable} \end{array}$$

Notes

- $H = \underline{S}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$. The map from K to H is nonlinear, so we have a nonlinear function of K to minimize.
- Instead, focus on the set of possible H.

Alternative formulation

Let $\mathcal{H}_{\text{rlzbl}} = \{\hat{H} \in RP \;;\; H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP\}$.

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H \in \mathcal{H}_{\text{rlzbl}} \\ & \text{The closed-loop is stable} \end{array}$$

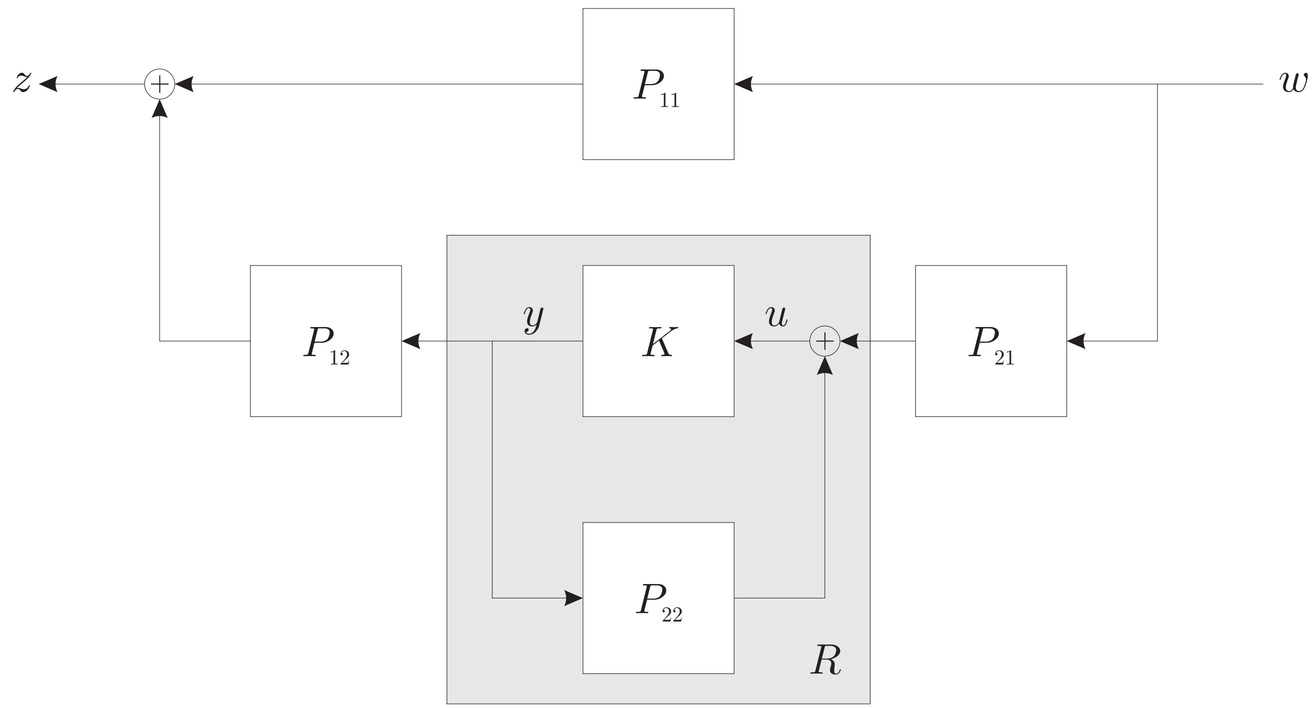
Theorem

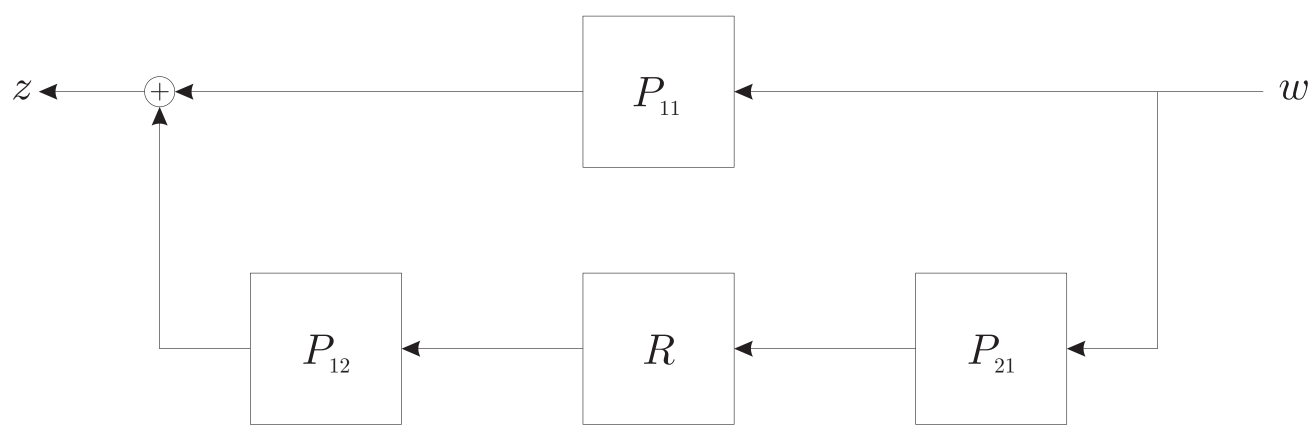
Suppose P_{22} is strictly proper. Then the set $\mathcal{H}_{\mathsf{rlzbl}}$ is affine.

Proof

- We know $H = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$. Let $R = (I - P_{22}K)^{-1}$. This map is one-to-one, since $K = (I + RP_{22})^{-1}R$, and $I + RP_{22}$ is always invertible (in RP). since P_{22} is strictly proper.
- So given $\hat{K} \in RP$ we can construct $\hat{R} \in RP$, and vice-versa. Hence
$$\mathcal{H}_{\mathsf{rlzbl}} = \left\{ P_{11} + P_{12}RP_{21} \ ; \ \hat{R} \in RP \right\}$$
- Suppose $H_a, H_b \in \mathcal{H}_{\mathsf{rlzbl}}$. We need to show that for any $\lambda \in \mathbb{R}$,
$$\lambda H_a + (1 - \lambda)H_b \in \mathcal{H}_{\mathsf{rlzbl}}$$
Let R_a and R_b be such that $H_a = P_{11} + P_{12}R_aP_{21}$ and $H_b = P_{11} + P_{12}R_bP_{21}$. Choose $R_\lambda = \lambda R_a + (1 - \lambda)R_b$. Then
$$\lambda H_a + (1 - \lambda)H_b = P_{11} + P_{12}R_\lambda P_{21}$$

Realizability





Realizability

The general optimization problem is

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H \in \mathcal{H}_{\mathsf{rlzbl}} \\ & \text{The closed-loop is stable} \end{array}$$

The set $\mathcal{H}_{\mathsf{rlzbl}}$ is

$$\mathcal{H}_{\mathsf{rlzbl}} = \left\{ P_{11} + P_{12} R P_{21} \ ; \ \hat{R} \in RP \right\}$$

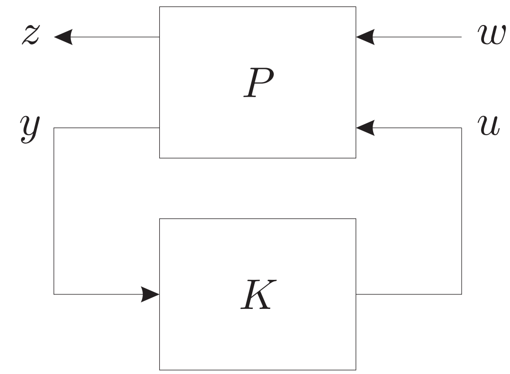
Equivalent problem

$$\begin{array}{rl} \text{minimize} & \|P_{11} + P_{12} R P_{21}\| \\ \text{subject to} & \text{The closed-loop is stable} \end{array}$$

Notes

- $\mathcal{H}_{\mathsf{rlzbl}}$ is convex, since it is affine.
- Optimization subject to the constraint that $H \in \mathcal{H}_{\mathsf{rlzbl}}$ may be tractable.
- Once R has been found, construct K from $K = (I + RP_{22})^{-1} R$.

Internal stability



The system interconnection is called *internally stable* if, for every initial condition $x(0)$ and $x_K(0)$,

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} x_K(t) = 0$$

when $w = 0$.

- We know

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x_K}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t)$$

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$

- The dynamics of the interconnected system are $\begin{bmatrix} \dot{x}(t) \\ \dot{x}_K(t) \end{bmatrix} = A_{\mathsf{cl}} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix}$ where

$$A_{\mathsf{cl}} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

- Hence the system is internally stable if and only if $I - D_{22}D_K$ is invertible and A_{cl} is Hurwitz.

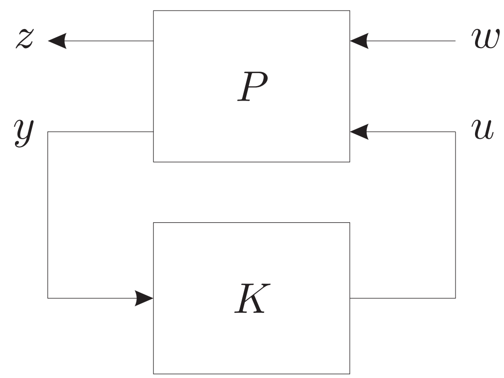
Internal stability

Suppose $\hat{P} \in RP$; that is $\hat{P}$ is a real-rational proper transfer function. Then

$$P \text{ is stable} \qquad \iff \qquad \hat{P} \in H_\infty$$

Exponential stability of the state then follows if the state-space realization for P is controllable and observable.

Linear Fractional Transformations



The map from w to z is given by

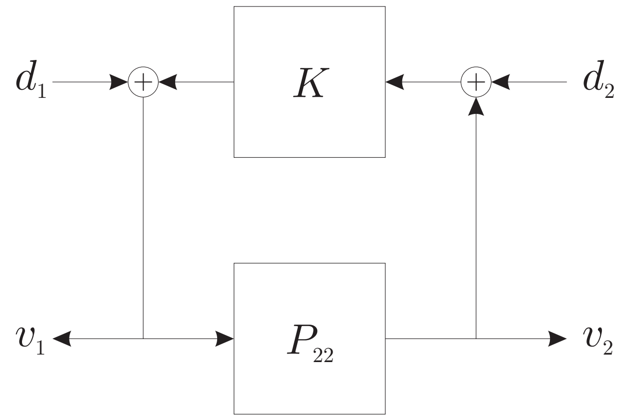
$$\underline{S}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

Is $\underline{S}(\hat{P}, \hat{K}) \in H_\infty$ equivalent to exponential stability of the states when the realizations of P and K are controllable and observable?

Answer: *No.* e.g pick $P_{12} = 0$.

Input-output characterization of internal stability

Consider the feedback loop:



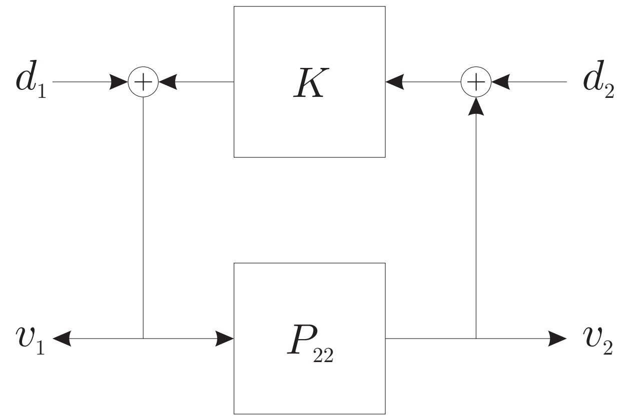
Inject *actuator and sensor noise* d_1 and d_2. Then

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = W \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \qquad \text{where} \qquad W = \begin{bmatrix} (I - KG)^{-1} & (I - KG)^{-1}K \\ G(I - KG)^{-1} & G(I - KG)^{-1}K \end{bmatrix}$$

- A state-space system is called *stabilizable* if for any initial condition $x(0)$ in the uncontrollable subspace, the state decays to zero.
- Similarly, a state-space system is called *detectable* if for any initial condition $x(0)$ in the unobservable subspace, the state decays to zero.

Suppose the realizations for P_{22} and K are stabilizable and detectable. Then the above feedback loop is internally stable if and only if $\hat{W} \in RH_\infty$.

Input-output characterization of internal stability



Suppose the realizations for P_{22} and K are stabilizable and detectable. Then the above feedback loop is internally stable if and only if

$$\begin{bmatrix} (I - KG)^{-1} & (I - KG)^{-1}K \\ G(I - KG)^{-1} & G(I - KG)^{-1}K \end{bmatrix} \in RH_\infty$$

- For scalar P_{22} and K, this is equivalent to the statement that there are no *unstable pole-zero cancellations*.
- The above definition is valid in the multivariable case also, when zeros are not clearly defined.
- Any sensible design problem would include signals d_1 and d_2 as part of w and signals v_1 and v_2 as part of z.

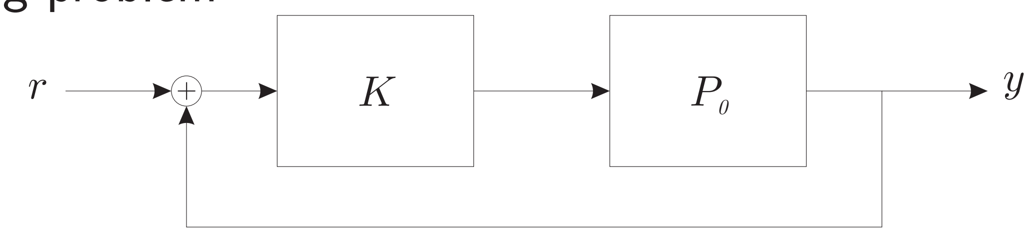
Example: unstable pole-zero cancellations

Consider the plant-controller pair

$$\hat{G}(s) = \frac{10 - s}{(s+10)s^2} \qquad \hat{K}(s) = \frac{-3(12+11s)}{10-s}$$

which has an unstable pole-zero cancellation.

Consider the tracking problem



Then

$$\hat{y}(s) = \frac{-3(12+11s)}{(s+4)(s+3)^2}\hat{r}(s)$$

But

$$W = \begin{bmatrix} \frac{(10+s)s^2}{(s+4)(s+3)^2} & \frac{3(12+11s)(10+s)s^2}{(s-10)(s+4)(s+3)^2} \\ \frac{10-s}{(s+4)(s+3)^2} & \frac{-3(12+11s)}{(s+4)(s+3)^2} \end{bmatrix}$$

and the pole-zero cancellation shows as instability of W.

Internal stability and LFTs

Suppose P and K are state-space systems with

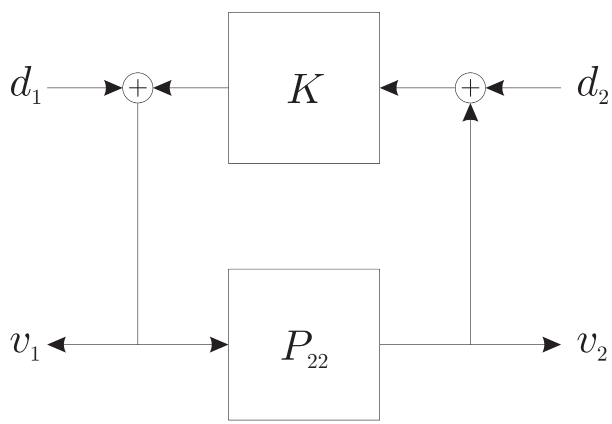
$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where} \quad \hat{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad \text{and} \quad \hat{P}_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$$

and

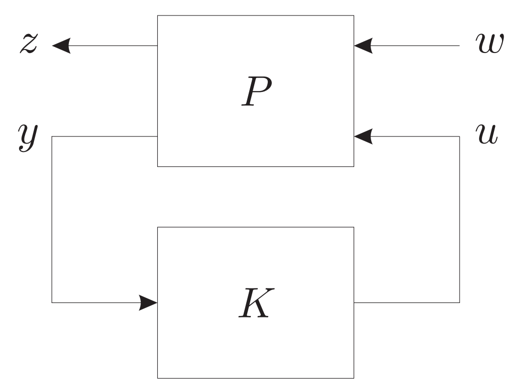
$$u = Ky \qquad \text{where} \qquad \hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Theorem

Suppose (A, B_2) is stabilizable and (A, C_2) is detectable. Then



$\Longleftrightarrow$



is internally stable — is internally stable