Engr210a Lecture 14: Youla parametrization

- Coprime factorization over RH_∞
- Internal stability
- Construction of a stabilizing controller
- Youla parametrization of all stabilizing controllers
- LFT formulation
- Affine parametrization of closed-loop map
- Optimization

Coprime factorization in RH_∞

Given $G \in RP$, there exist both left and right coprime factorizations.

- *Right:* There exist $N_r, M_r, X_r, Y_r \in RH_\infty$ such that N_r, M_r are right-coprime, and

$$G = N_r M_r^{-1} \qquad \text{and} \qquad X_r M_r - Y_r N_r = I$$

- *Left:* There exist $N_l, M_l, X_l, Y_l \in RH_\infty$ such that N_l, M_l are left-coprime, and

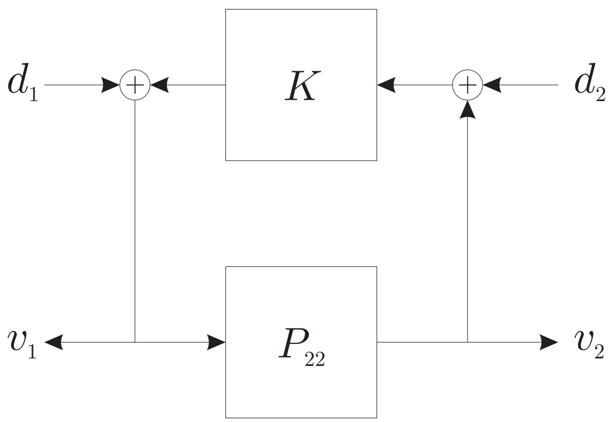
$$G = M_l^{-1} N_l \qquad \text{and} \qquad M_l X_l - N_l Y_l = I$$

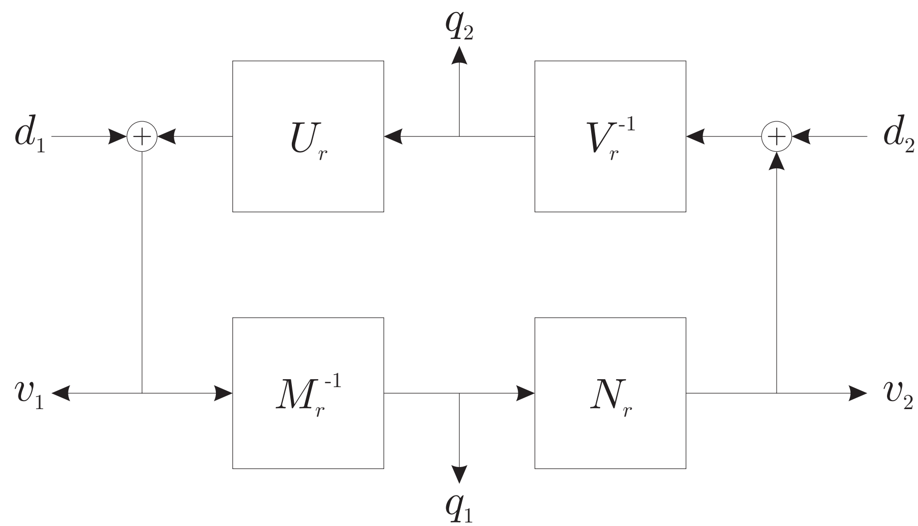
Double coprime factorization

Given $G \in RP$, there exist left and right coprime factorizations, with the additional property that

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} = I$$

Coprime factorization and internal stability





Define the maps

$$W : \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{and} \qquad S : \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

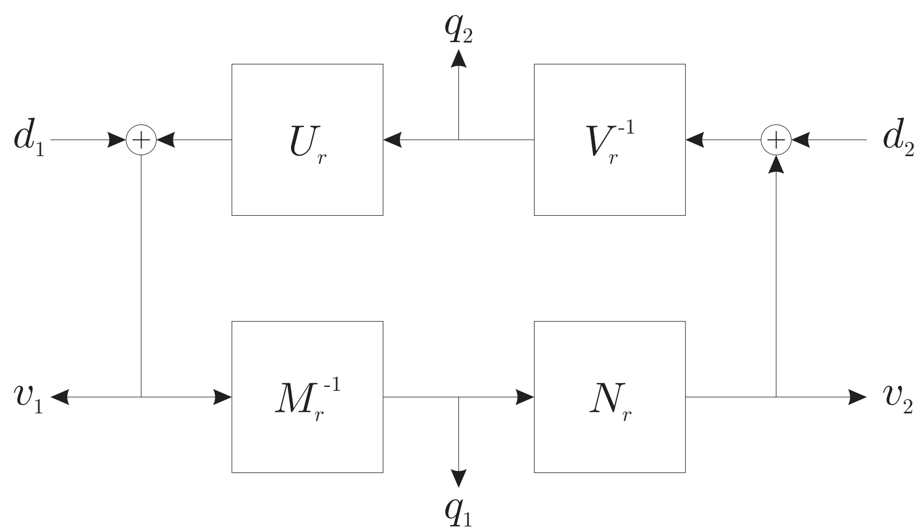
Theorem

Suppose $P_{22}, K \in RP$ and $P_{22} = N_r M_r^{-1}$, $K = U_r V_r^{-1}$ are right coprime factorizations. Then

$$W \in RH_\infty \qquad \iff \qquad S \in RH_\infty$$

Theorem

$$W : \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in RH_\infty \qquad \Longleftrightarrow \qquad S : \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in RH_\infty$$



Proof: $\Longleftarrow$

- $$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} M_r & -U_r \\ -N_r & V_r \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \Longrightarrow \qquad S = \begin{bmatrix} M_r & -U_r \\ -N_r & V_r \end{bmatrix}^{-1}$$
- $$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} M_r & 0 \\ N_r & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \Longrightarrow \qquad W = \begin{bmatrix} M_r & 0 \\ N_r & 0 \end{bmatrix} S$$
- Hence $S \in RH_\infty$ implies $W \in RH_\infty$.

Proof: $\Longrightarrow$

- $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} M_r \\ N_r \end{bmatrix} q_1$, and the Bezout equation for P_{22} is $\begin{bmatrix} X_r & -Y_r \end{bmatrix} \begin{bmatrix} M_r \\ N_r \end{bmatrix} = I$.

  Hence $q_1 = \begin{bmatrix} X_r & -Y_r \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} X_r & -Y_r \end{bmatrix} W \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

- Similarly $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} U_r \\ V_r \end{bmatrix} q_2 + \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, and the Bezout equation for K is

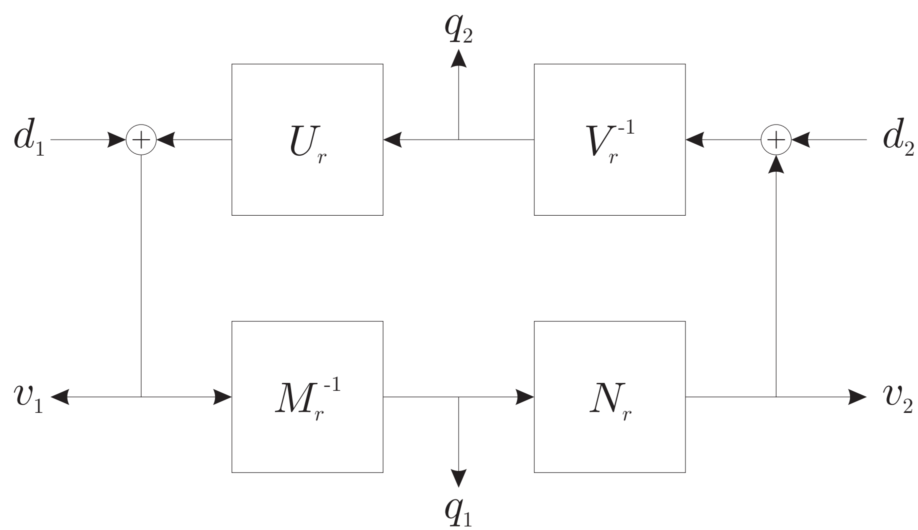
  $\begin{bmatrix} X_r^K & -Y_r^K \end{bmatrix} \begin{bmatrix} U_r \\ V_r \end{bmatrix} = I$.

  Hence $q_2 = \begin{bmatrix} X_r^K & -Y_r^K \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} X_r^K & Y_r^K \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

- Hence $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} X_r & -Y_r \\ X_r^K & Y_r^K \end{bmatrix} \left(W - \begin{bmatrix} 0 & 0 \\ X_r^K & Y_r^K \end{bmatrix} \right) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

- Hence $S = \begin{bmatrix} X_r & -Y_r \\ X_r^K & Y_r^K \end{bmatrix} \left(W - \begin{bmatrix} 0 & 0 \\ X_r^K & Y_r^K \end{bmatrix} \right)$ and so if $W \in RH_\infty$ then $S \in RH_\infty$.

Coprime factorization and internal stability



Notes

- The controller K is internally stabilizing if and only if $S = \begin{bmatrix} M_r & -U_r \\ -N_r & V_r \end{bmatrix}^{-1} \in RH_\infty$.

- Similarly for left coprime factorizations. The controller $K = V_l^{-1} U_l$ is internally stabilizing if and only if $\begin{bmatrix} V_l & -U_l \\ -N_l & M_l \end{bmatrix}^{-1} \in RH_\infty$.

Stabilizing controllers

Suppose doubly coprime factorizations of P_{22} are $P_{22} = N_r M_r^{-1} = M_l^{-1} N_l$ where

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} = I$$

Given $Q \in RH_\infty$, let

$$U_r = Y_l - M_r Q \qquad U_l = Y_r - Q M_l$$

$$V_r = X_l - N_r Q \qquad V_l = X_r - Q N_l$$

Then if V_r and V_l are invertible in RP,

$$U_r V_r^{-1} = V_l^{-1} U_l$$

are right and left coprime factorizations of a stabilizing controller K.

Notes: If $Q = 0$, then internal stability is immediate, since

$$\begin{aligned} \begin{bmatrix} M_r & -U_r \\ -N_r & V_r \end{bmatrix}^{-1} &= \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix}^{-1} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix}^{-1} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

which is in RH_∞.

Proof

First we prove that $U_r V_r^{-1} = V_l^{-1} U_l$.

- Doubly coprime factorization implies

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} = I$$

which implies

$$\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} = I$$

- Expanding this gives

$$\begin{bmatrix} X_r - QN_l & -(Y_r - QM_l) \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l - M_r Q \\ N_r & X_l - N_r Q \end{bmatrix} = I$$

- The $(1,2)$ block of this equation is

$$(X_r - QN_l)(Y_l - M_r Q) - (Y_r - QM_l)(X_l - N_r Q) = 0$$

which is

$$V_l U_r = U_l V_r \qquad \text{which implies} \qquad U_r V_r^{-1} = V_l^{-1} U_l$$

Proof

Now we prove that K is stabilizing.

- As before, doubly coprime factorization implies

$$\begin{bmatrix} X_r - QN_l & -(Y_r - QM_l) \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l - M_rQ \\ N_r & X_l - N_rQ \end{bmatrix} = I$$

which is just

$$\begin{bmatrix} V_l & -U_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix} = I$$

- Hence

$$S = \begin{bmatrix} M_r & -U_r \\ -N_r & V_r \end{bmatrix}^{-1} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix}^{-1} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} V_l & -U_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

and hence $S \in RH_\infty$.

- Finally, to show coprimeness, note that this implies the two Bezout equations

$$M_lV_r - N_lU_r = I \qquad \text{and} \qquad V_lM_r - U_lN_r = I$$

which implies U_r, V_r are right coprime and U_l, V_l are left coprime.

Theorem

The controller $K \in RP$ is stabilizing if and only if there exists $Q \in RH_\infty$ such that

$$K = U_r V_r^{-1} \qquad \text{where} \qquad \begin{aligned} U_r &= Y_l - M_r Q \\ V_r &= X_l - N_r Q \end{aligned}$$

Notes

- This result is called the *Youla Parametrization*.
- Every stabilizing controller has the above form.
- We have already proved the *if* direction; all that remains is *only if*.

Proof

- Suppose K is stabilizing, and let $K = U_r V_r^{-1}$ be a right coprime factorization.
- Doubly coprime factorization implies

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} = I$$

which implies

$$\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix} = \begin{bmatrix} I & X_r U_r - Y_r V_r \\ 0 & \Theta \end{bmatrix}$$

where $\Theta = M_l V_r - N_l U_r$. Let $Q = -(X_r U_r - Y_r V_r)\Theta^{-1}$.

- $\begin{bmatrix} X_r & -Y_r \\ -N_l & M_l \end{bmatrix}^{-1} = \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} \in RH_\infty$ from the doubly coprime factorization.

  $\begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix}^{-1} \in RH_\infty$ since the system is internally stable. Hence $\Theta^{-1} \in RH_\infty$.

- Hence $\begin{bmatrix} M_r & U_r \\ N_r & V_r \end{bmatrix} = \begin{bmatrix} M_r & Y_l \\ N_r & X_l \end{bmatrix} \begin{bmatrix} I & X_r U_r - Y_r V_r \\ 0 & \Theta \end{bmatrix} = \begin{bmatrix} M_r & (Y_l - M_r Q)\Theta \\ N_r & (X_l - N_r Q)\Theta \end{bmatrix}$
- $Q \in RH_\infty$, and the controller $K = U_r V_r^{-1} = (Y_l - M_r Q)(X_l - N_r Q)^{-1}$ as required.

LFT form of all stabilizing controllers

We have

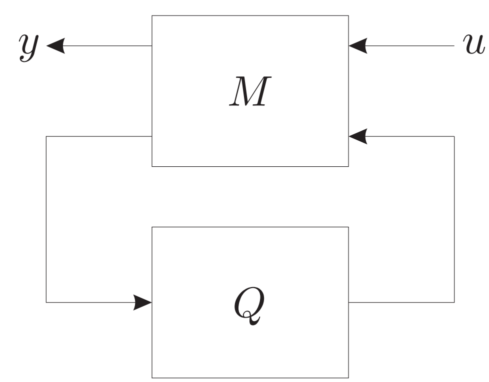
$$(A+BQ)(C+DQ)^{-1} = \underline{S}(M,Q) \qquad \text{where} \qquad M = \begin{bmatrix} AC^{-1} & B - AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix}$$

Hence for $K = (Y_l - M_r Q)(X_l - N_r Q)^{-1}$, we have

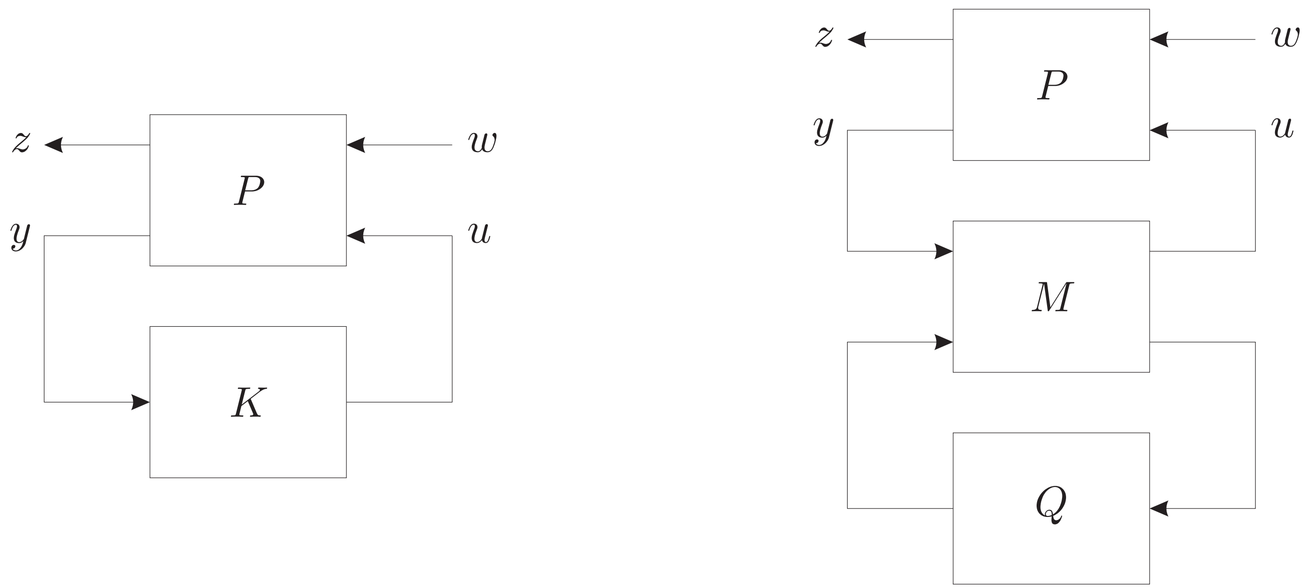
$$K = \underline{S}(M,Q) \qquad \text{where} \qquad M = \begin{bmatrix} Y_l X_l^{-1} & -X_r^{-1} \\ X_l^{-1} & X_l^{-1} N_r \end{bmatrix}$$

where we have used the fact that $Y_l X_l^{-1} = X_r Y_r^{-1}$.

Hence every stabilizing controller has the form



The closed-loop system



The closed-loop map $H = \underline{S}(P, K)$ is given by

$$H = T_1 + T_2 Q T_3$$

where

$$\begin{aligned} T_1 &= P_{11} + P_{12} Y_l M_l P_{21} \\ T_2 &= P_{12} M_r \\ T_3 &= M_l P_{21} \end{aligned}$$

Further, $T_1, T_2, T_3 \in RH_\infty$.

General problem

The general problem is

$$\begin{array}{rl} \text{minimize} & \|H\| \\ \text{subject to} & H = \underline{S}(P, K) \text{ for some } \hat{K} \in RP \\ & \text{The closed-loop is stable} \end{array}$$

Equivalent formulation

$$\begin{array}{rl} \text{minimize} & \|T_1 + T_2 Q T_3\| \\ \text{subject to} & Q \in H_\infty \end{array}$$

Once the optimal Q is found, then the optimal K is given by

$$K = (Y_l - M_r Q)(X_l - N_r Q)^{-1}$$