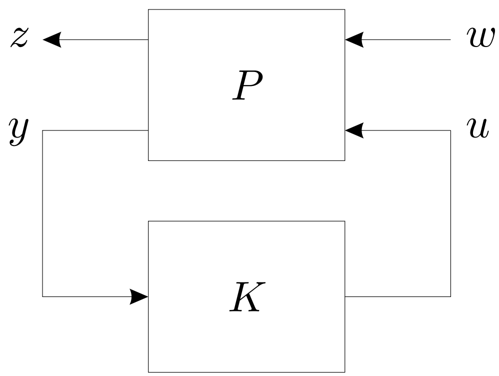
Engr210a Lecture 16: H_∞ synthesis

- State-feedback problem
- KYP formulation
- Change of variables
- Synthesis theorem
- General problem formulation
- Change of variables
- Synthesis theorem
- Formal correspondence
- The H_2 problem

Problem formulation

We have the following interconnection



and would like to find K to solve

$$\begin{array}{rl} \text{minimize} & \|\underline{S}(P,K)\| \\ \text{subject to} & \text{The closed-loop is stable} \end{array}$$

State feedback problem

We are given the plant P

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$$
$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t)$$

Controller

Find a controller K of the form

$$u(t) = D_K x(t)$$

Closed-loop equations

We are given the plant P

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \end{aligned}$$

and controller $u = D_K x$.

The closed-loop map from w to z is

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl} x_{cl}(t) + B_{cl} w(t) \\ z(t) &= C_{cl} x(t) + D_{cl} w(t) \end{aligned}$$

where

$$\begin{aligned} A_{cl} &= A + B_2 D_K & B_{cl} &= B_1 \\ C_{cl} &= C_1 + D_{12} D_K & D_{cl} &= D_{11} \end{aligned}$$

Convenient form

Closed-loop parameters are affine in controller parameters:

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} 0 \\ D_K \end{bmatrix}$$

The KYP lemma

The following are equivalent

- $\|G\| < 1$ and A is Hurwitz.
- There exists $X \in \mathbb{R}^{n\times n}$ such that

$$X > 0 \qquad \text{and} \qquad \begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} + \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Schur complement

Applying the Schur complement gives the equivalent statement

- There exists $X \in \mathbb{R}^{n\times n}$ such that

$$X > 0 \qquad \text{and} \qquad \begin{bmatrix} A^*X + XA & XB & C^* \\ B^*X & -I & D^* \\ C & D & -I \end{bmatrix} < 0$$

The KYP Lemma

The following are equivalent

- $\|\underline{S}(P,K)\| < 1$ and A_{cl} is Hurwitz.
- There exists $X_{cl} \in \mathbb{R}^{n\times n}$ such that

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} \\ B_{cl}^* X_{cl} & -I \end{bmatrix} + \begin{bmatrix} C_{cl}^* \\ D_{cl}^* \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

Schur complement

Applying the Schur complement gives the equivalent statement

- There exists $X_{cl} \in \mathbb{R}^{n\times n}$ such that

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

Convexity

We would like to find D_K such that

$$\text{there exists } X_{cl} > 0 \text{ such that } \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

We know $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} 0 \\ D_K \end{bmatrix}$.

Hence the above conditions are

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} (A + B_2 D_K)^* X_{cl} + X_{cl}(A + B_2 D_K) & X B_1 & C_1 + D_{12} D_K \\ B_1^* X & -I & D_{11} \\ C_1^* + D_K^* D_{12}^* & D_{11}^* & -I \end{bmatrix} < 0$$

Notes

- If we know X_{cl}, then above inequality is affine in D_K. Hence it is an LMI and we can find the controller.
- If we know the controller parameter D_K, then the inequality is affine in X_{cl}, and we can compute the closed-loop norm.
- Both at once?

Transformation

Let $Y_{cl} = \left(X_{cl}^{-1}\right)^*$ and $C_n = D_K Y_{cl}$.

Define

The following is an affine function of C_n and Y_{cl}.

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} AY_{cl} & B_1 \\ C_1 Y_{cl} & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} 0 \\ C_n \end{bmatrix}$$

Lemma

The following will be useful in transforming the KYP lemma

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} Y_{cl}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{cl} A_{cl} & X_{cl} B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

Proof

Substitute

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} 0 \\ D_K \end{bmatrix}$$

Synthesis theorem

The following are equivalent

- There exists a stabilizing state-feedback controller such that $\|\underline{S}(P,K)\| < 1$
- There exists $Y_{cl} > 0$ and C_n such that

$$\begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

Notes

- Recall $\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} AY_{cl} & B_1 \\ C_1 Y_{cl} & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} 0 \\ C_n \end{bmatrix}$
- This is affine in Y_{cl} and C_n.
- Hence the above inequalities are LMIs in these variables.
- Once we have found Y_{cl} and C_n, we can find D_K from the formula

$$D_K = C_n Y_{cl}^{-1}$$

Proof

- A stabilizing controller which achieves $\|\underline{S}(P,K)\| < 1$ exists iff

  $$\text{there exists } X_{cl} > 0 \text{ such that} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

- This is equivalent to

  $$\text{there exists } Y_{cl} > 0 \text{ s.t.} \quad \begin{bmatrix} Y_{cl}^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

  Note invertibility of $Y_{cl} = \left(X_{cl}^{-1}\right)^*$.

- This inequality is equivalent to

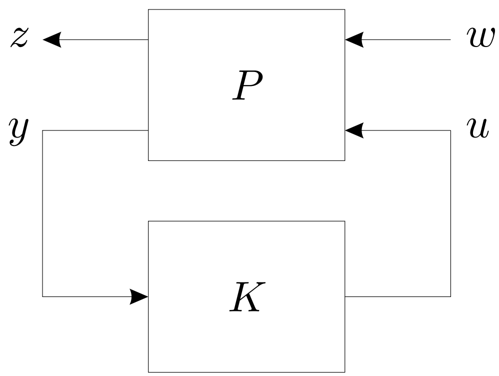
  $$\begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

  where we have used

  $$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} Y_{cl}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{cl} A_{cl} & X_{cl} B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

Problem formulation

We have the following interconnection



and would like to find K to solve

$$\begin{array}{ll} \text{minimize} & \|\underline{S}(P,K)\| \\ \text{subject to} & \text{The closed-loop is stable} \end{array}$$

State-space

We are given the plant P

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

which is not necessarily stable.

In state-space

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{aligned}$$

Controller

Find a controller K of the form

$$\begin{aligned} \dot{x}_K(t) &= A_K x(t) + B_K y(t) \\ u(t) &= C_K x(t) + D_K y(t) \end{aligned}$$

Controller order

- We start by assuming the controller has order $n_K \geq n$.
- We will find that any performance achievable by a controller of order $n_K > n$ is also achievable by one of order n.

Assumption

- *Direct feedthrough terms*: Assume that $D_{22} = 0$; if $D_{22} \neq 0$ the same approach works with a simple change of variables.

Closed-loop equations

The closed-loop map from w to z is

$$\begin{aligned} \dot{x}_{cl}(t) &= A_{cl}x_{cl}(t) + B_{cl}w(t) \\ z(t) &= C_{cl}x(t) + D_{cl}w(t) \end{aligned}$$

Closed-loop parameters are affine in controller parameters:

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

The KYP Lemma

The following are equivalent

- $\|\underline{S}(P,K)\| < 1$ and A_{cl} is Hurwitz.
- There exists $X_{cl} \in \mathbb{R}^{(n+n_K)\times(n+n_K)}$ such that

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} \\ B_{cl}^* X_{cl} & -I \end{bmatrix} + \begin{bmatrix} C_{cl}^* \\ D_{cl}^* \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

Schur complement

Applying the Schur complement gives the equivalent statement

- There exists $X_{cl} \in \mathbb{R}^{(n+n_K)\times(n+n_K)}$ such that

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

Convexity

We would like to find A_K, B_K, C_K, D_K such that

$$\text{there exists } X_{cl} > 0 \text{ such that } \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

We know

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

Notes

- If we know X_{cl}, then above inequality is affine in A_K, B_K, C_K, D_K, hence it is an LMI and we can find the controller.
- If we know the controller A_K, B_K, C_K, D_K, then the inequality is affine in X_{cl}, and we can compute the closed-loop norm.
- Both at once?

Transformation formulae

Given X_{cl} and A_K, B_K, C_K, D_K define the following new variables

- Define X, X_2 and Y, Y_2 by

$$X_{cl} = \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \qquad X_{cl}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$$

- Define A_n, B_n, C_n, D_n by

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} X_2 & XB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} Y_2^* & 0 \\ C_2 Y & I \end{bmatrix} + \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix}$$

- Define Y_{cl} by

$$Y_{cl} = \begin{bmatrix} Y & I \\ Y_2^* & 0 \end{bmatrix}$$

Notes

- $X_{cl} > 0$ implies $X > 0$ and $Y > 0$.
- Dimensions: $X, Y \in \mathbb{R}^{n \times n}$, $X_2, Y_2 \in \mathbb{R}^{n \times n_K}$ and $Y_{cl} \in \mathbb{R}^{(n+n_K) \times 2n}$.
- We will write the KYP LMI for the closed-loop in terms of X, Y, A_n, B_n, C_n, D_n.

Transformation lemma

Given $X, Y \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$, then

- There exist $X_2, X_3, Y_2, Y_3 \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$
- Let X_{cl} and Y_{cl} be as above. Then $X_{cl} > 0$ and Y_{cl} has full column rank.

Proof

- Since $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$, we have $I - XY$ is nonsingular.
- We need to find X_2 and Y_2 such that

$$X_2 Y_2^* = I - XY$$

  Here X_2 and Y_2 can be chosen square and nonsingular. (e.g. choose $Y_2 = I$.)

- Then X_{cl} is uniquely determined, since

$$\begin{bmatrix} Y & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & X_2 \end{bmatrix} \quad \Longrightarrow \quad X_{cl} = \begin{bmatrix} Y & Y_2 \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ X & X_2 \end{bmatrix}$$

- Y_{cl} has full column rank, since Y_2 is nonsingular, and $Y_{cl} = \begin{bmatrix} Y & I \\ Y_2^* & 0 \end{bmatrix}$

Converse transformation lemma

Given $X_{cl} > 0$ such that X_2 has full row rank. Then

- $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} > 0$
- Y_{cl} has full column rank.

Proof

- Clearly $\begin{bmatrix} I & 0 \\ X & X_2 \end{bmatrix}$ has full row rank.
- Note that $\begin{bmatrix} Y & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & X_2 \end{bmatrix}$ and hence

$$Y_{cl} = \begin{bmatrix} Y & I \\ Y_2^* & 0 \end{bmatrix} = \begin{bmatrix} I & X^* \\ 0 & X_2^* \end{bmatrix} X_{cl}^{-1}$$

  has full column rank.
- $\begin{bmatrix} Y & I \\ I & X \end{bmatrix} = Y_{cl}^* X_{cl} Y_{cl}$, which is positive definite since Y_{cl} has full column rank.

Definitions

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} AY & A & B_1 \\ 0 & XA & XB_1 \\ C_1Y & C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{bmatrix}$$

$$X_v = \begin{bmatrix} Y & I \\ I & X \end{bmatrix}$$

Lemma

$$\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} Y_{cl}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{cl}A_{cl} & X_{cl}B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad X_v = Y_{cl}^* X_{cl} Y_{cl}$$

Proof: Use the following previous definitions

- $Y_{cl} = \begin{bmatrix} Y & I \\ Y_2^* & 0 \end{bmatrix}$
- $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$
- $\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} X_2 & XB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} Y_2^* & 0 \\ C_2Y & I \end{bmatrix} + \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix}$

Theorem

There exists a stabilizing controller K such that

$$\|\underline{S}(P,K)\| < 1$$

if and only if there exists X, Y, A_n, B_n, C_n, D_n satisfying the LMIs

$$X_v > 0 \quad \text{and} \quad \begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

Notes

- The above inequality is affine, and hence an LMI, in X, Y, A_n, B_n, C_n, D_n.
- Once we have found X, Y, A_n, B_n, C_n, D_n, we can find X_2, Y_2 from

  $$X_2 Y_2^* = I - XY$$

  and hence find the controller using

  $$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} X_2 & XB_2 \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} Y_2^* & 0 \\ C_2 Y & I \end{bmatrix}^{-1}$$

- We can always find a controller of dimension n, since X_2 can be chosen to be $n \times n$.

Proof (only if)

Suppose there exists X_{cl} and A_K, B_K, C_K, D_K satisfying

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

then since the set of such X_{cl} is open, we can perturb X_{cl} so that $X_{cl} = \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}$ satisfies the above inequalities with X_2 having full row rank.

Then according to the converse transformation lemma, Y_{cl} has full column rank, and hence

$$Y_{cl}^* X_{cl} Y_{cl} > 0 \quad \text{and} \quad \begin{bmatrix} Y_{cl}^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

These inequalities are

$$X_v > 0 \quad \text{and} \quad \begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

since we know $\begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} = \begin{bmatrix} Y_{cl}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{cl} A_{cl} & X_{cl} B \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$.

Proof (if)

Suppose there exist X and Y such that

$$X_v > 0 \quad \text{and} \quad \begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

Then, from the transformation lemma, we can construct $X_{cl} > 0$ and Y_{cl} with full row rank which satisfy

$$Y_{cl}^* X_{cl} Y_{cl} > 0 \quad \text{and} \quad \begin{bmatrix} Y_{cl}^* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

hence

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

for A_K, B_K, C_K, D_K defined by

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} X_2 & X B_2 \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} Y_2^* & 0 \\ C_2 Y & I \end{bmatrix}^{-1}$$

Formal correspondence

Compare the KYP LMIs

$$X_{cl} > 0 \qquad \text{and} \qquad \begin{bmatrix} A_{cl}^* X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^* \\ B_{cl}^* X_{cl} & -I & D_{cl}^* \\ C_{cl} & D_{cl} & -I \end{bmatrix} < 0$$

with the H_∞ synthesis LMIs

$$X_v > 0 \quad \text{and} \quad \begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -I & D_v^* \\ C_v & D_v & -I \end{bmatrix} < 0$$

Optimizing the induced-norm

The following problem

$$\begin{aligned} \text{minimize} \quad & \|\underline{S}(P,K)\| \\ \text{subject to} \quad & \text{The closed-loop is stable} \end{aligned}$$

is equivalent to the LMI problem

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to} \quad & X_v > 0 \\ & \begin{bmatrix} A_v^* + A_v & B_v & C_v^* \\ B_v^* & -\gamma I & D_v^* \\ C_v & D_v & -\gamma I \end{bmatrix} < 0 \end{aligned}$$

Notes

- No bisection search necessary.
- The LMI variables are $\gamma, X, Y, A_n, B_n, C_n, D_n$.

The H_2 problem

Given the LTI system

$$\dot{x}(t) = Ax(t) + Bw(t) \qquad x(0) = 0$$
$$z(t) = Cx(t) + Dw(t)$$

The H_2 norm of G is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\big(\hat{G}^*(j\omega)\hat{G}(j\omega)\big)\, d\omega$$

where

$$\hat{G}(j\omega) = C(j\omega I - A)^{-1}B + D$$

H_2 analysis

The following are equivalent

- $\|G\|_2 \le \gamma$
- there exist X and Z such that

$$D = 0, \qquad \begin{bmatrix} A^*X + XA & XB \\ B^*X & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} X & C^* \\ C & Z \end{bmatrix} > 0, \qquad \mathrm{Trace}(Z) < \gamma$$

H_2 synthesis

There exists a stabilizing controller K such that

$$\|\underline{S}(P, K)\|_2 < \gamma$$

if and only if there exist $Z, X, Y, A_n, B_n, C_n, D_n$ satisfying the LMIs

$$D_v = 0, \qquad \begin{bmatrix} A_v^* + A_v & B_v \\ B_v^* & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} X_v & C_v^* \\ C_v & Z \end{bmatrix} > 0, \qquad \mathrm{Trace}(Z) < \gamma$$