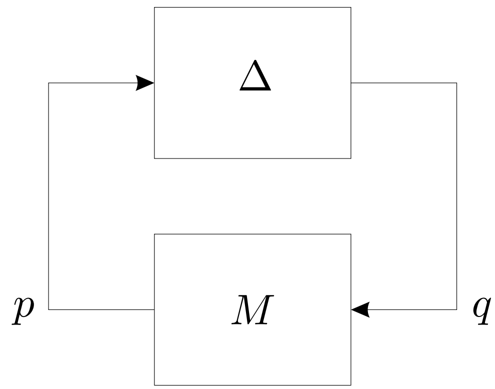
Engr210a Lecture 18: The structured singular value

- Structure specifications
- LTI uncertainty
- Parametric uncertainty
- The structured singular value
- Upper and lower bounds
- The matrix structured singular value
- Computation

Structure specifications



Define the set

$$\mathbf{C\Delta} = \left\{ \mathrm{diag}(\Delta_1, \ldots, \Delta_d) \ ; \ \Delta_i \in \mathcal{L}(L_2) \right\}$$

The set $\mathbf{C\Delta}$ is called a *structure specification*.

Notes

- $\mathbf{C\Delta}$ is a convex cone.

Linear time-invariant uncertainty

One common structure specification is

$$\mathbf{C\Delta_{TI}} = \Big\{ \Delta = \operatorname{diag}(\Delta_1, \dots, \Delta_d) \ ; \ \Delta \in \mathcal{L}(L_2), \Delta \text{ is LTI} \Big\}$$

The commutant set is

$$\mathbf{\Theta_{TI}} = \Big\{ \Theta \in \mathcal{L}(L_2) \ ; \ \Theta \text{ is nonsingular and LTI}, \hat{\Theta}(s) = \operatorname{diag}\big(\hat{\theta}_1(s) I, \dots, \hat{\theta}_d(s) I\big) \Big\}$$

Interpretation

- Δ is linear time-invariant uncertainty.
- Δ represents *unmodeled* dynamics. Examples include
    - model reduction
    - modeling assumptions; e.g. rigidity in a structure
    - spatial discretization of continuum mechanics
- Commutant contains only LTI operators, since the delay is in $\mathbf{C\Delta_{TI}}$.

Parametric uncertainty

Consider the set of matrices

$$\mathbf{C\Delta}_{\mathsf{s,f}} = \Big\{ \Delta = \operatorname{diag}\big(\delta_1 I_{m_1}, \ldots, \delta_s I_{m_s}, \Delta_{s+1}, \ldots, \Delta_{s+f}\big) \ ; \ \delta_i \in \mathbb{C},\ \Delta_k \in \mathbb{C}^{m_k \times m_k} \Big\}$$

Interpretation

- $s =$ no. of scalar blocks

  $f =$ no. of full blocks
- Δ represents unknown parameters in the system.

Perturbation notation

$$\mathbf{C\Delta}_{\mathsf{s,f}} = \Big\{ \Delta = \operatorname{diag}\big(\delta_1 I_{m_1}, \ldots, \delta_s I_{m_s}, \Delta_{s+1}, \ldots, \Delta_{s+f}\big) \ ; \ \delta_k \in \mathbb{C},\ \Delta_k \in \mathbb{C}^{m_k \times m_k} \Big\}$$

$$\mathbf{\Delta}_{\mathsf{s,f}} = \Big\{ \Delta \in \mathbf{C\Delta}_{\mathsf{s,f}} \ ; \ \|\Delta\| \le 1 \Big\}$$

Commutant notation

$$\mathbf{\Theta}_{\mathsf{s,f}} = \Big\{ \operatorname{diag}\big(\Theta_1, \ldots, \Theta_s, \theta_{s+1} I, \ldots, \theta_{s+f} I\big) \text{ nonsingular} \ ; \ \Theta_k \in \mathbb{C}^{m_k \times m_k},\ \theta_k \in \mathbb{C} \Big\}$$

$$\mathbf{P\Theta}_{\mathsf{s,f}} = \Big\{ \Theta \in \mathbf{\Theta}_{\mathsf{s,f}} \ ; \ \Theta = \Theta^*,\ \Theta > 0 \Big\}$$

The structured singular value

Define the following function

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf\Big\{ \|\Delta\| \ ; \ \Delta \in \mathbf{C\Delta}, \ I - M\Delta \text{ is singular} \Big\}}$$

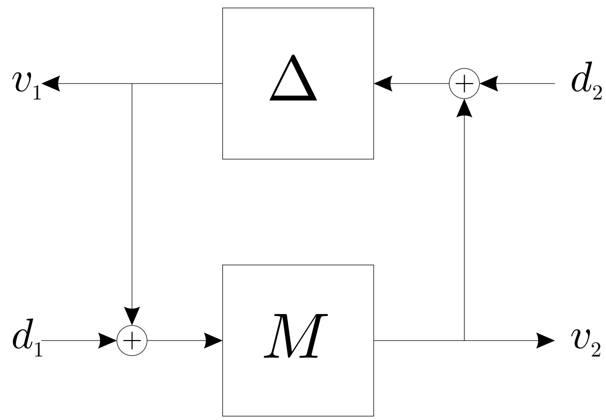
Interpretation

- $\mu(M, \mathbf{\Delta}) = \dfrac{1}{\inf\{\text{norm of destabilizing perturbations in } \mathbf{C\Delta}\}}$
- $\mu(M, \mathbf{\Delta})$ depends on the operator M and the structure specification $\mathbf{C\Delta}$.

Properties

- $\mu(\alpha M, \mathbf{\Delta}) = |\alpha| \mu(M, \mathbf{\Delta})$ for all $\alpha \in \mathbb{C}$.
- For general structure specifications $\mathbf{\Delta}$, $\mu(M, \mathbf{\Delta})$ is not a norm on M, since it does not satisfy the triangle inequality.

Stability



$$Z = \begin{bmatrix} I & -M \\ -\Delta & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - M\Delta)^{-1} & M(I - \Delta M)^{-1} \\ (I - \Delta M)^{-1} & (I - \Delta M)^{-1} \end{bmatrix}$$

Definitions

- If Δ, M are state-space systems, then the loop is called *internally-stable* if the states tend to zero.
- If $\Delta, M \in RH_\infty$, then the loop is called *input-output stable* if $Z \in RH_\infty$.

  This is equivalent to internal stability for stabilizable and detectable M, Δ.
- If $\Delta, M \in \mathcal{L}(L_2)$ then the loop is called *well-connected* if $Z \in \mathcal{L}(L_2)$.

The structured singular value

Suppose $\mathbf{C\Delta} \subseteq \mathcal{L}(L_2)$ is a convex cone. Define

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf\Big\{ \|\Delta\| \;;\; \Delta \in \mathbf{C\Delta},\; I - M\Delta \text{ is singular} \Big\}}$$

Recall the norm-bounded perturbation set

$$\mathbf{\Delta} = \Big\{ \Delta \in \mathbf{C\Delta} \;;\; \|\Delta\| \le 1 \Big\}$$

Robust well-connectedness

Sufficient condition

$$\mu(M, \mathbf{\Delta}) < 1 \qquad \Longrightarrow \qquad I - M\Delta \text{ is nonsingular for all } \Delta \in \mathbf{\Delta}$$

Necessary condition

$$\mu(M, \mathbf{\Delta}) \le 1 \qquad \Longleftarrow \qquad I - M\Delta \text{ is nonsingular for all } \Delta \in \mathbf{\Delta}$$

One scalar uncertainty block

Consider the uncertainty ball

$$\mathbf{\Delta}_{\mathsf{scalar}} = \Big\{ \delta I \; ; \; \delta \in \mathbb{C}, \; |\delta| \leq 1 \Big\}$$

We have

$$\begin{aligned} \mu(M, \mathbf{\Delta}_{\mathsf{scalar}}) &= \frac{1}{\inf \Big\{ |\delta| \; ; \; \delta \in \mathbb{C}, \; I - M\delta \text{ is singular} \Big\}} \\ &= \sup \Big\{ |\delta^{-1}| \; ; \; \delta \in \mathbb{C}, \; \delta^{-1} I - M \text{ is singular} \Big\} \\ &= \sup \Big\{ |\lambda| \; ; \; \lambda \in \mathbb{C}, \; \lambda I - M \text{ is singular} \Big\} \\ &= \rho(M) \end{aligned}$$

For uncertainty with one scalar uncertainty block, the structure singular value is equal to the spectral radius.

One full uncertainty block

Consider the uncertainty ball

$$\mathbf{\Delta}_{\mathsf{full}} = \Big\{ \Delta \; ; \; \Delta \in \mathcal{L}(L_2), \; \|\Delta\| \leq 1 \Big\}$$

We have

$$\begin{aligned} \mu(M, \mathbf{\Delta}_{\mathsf{full}}) &= \frac{1}{\inf\Big\{ \|\Delta\| \; ; \; \Delta \in \mathbf{C}\mathbf{\Delta}_{\mathsf{full}}, \; I - M\Delta \text{ is singular} \Big\}} \\ &= \sup\Big\{ \|\Delta\|^{-1} \; ; \; \Delta \in \mathcal{L}(L_2), \; I - M\Delta \text{ is singular} \Big\} \\ &= \|M\| \end{aligned}$$

For uncertainty with one full uncertainty block, the structure singular value is equal to the operator norm.

General uncertainty

- For any uncertainty specification, $\mathbf{\Delta}_{\mathsf{scalar}} \subseteq \mathbf{\Delta} \subseteq \mathbf{\Delta}_{\mathsf{full}}$.
- Hence $\rho(M) \leq \mu(M, \mathbf{\Delta}) \leq \|M\|$.

An upper-bound for the structured singular value

Commutant property

For any $\Delta \in \mathbf{\Delta}$, we have

$$\Theta\Delta = \Delta\Theta \text{ for all } \Theta \in \mathbf{\Theta}$$

Hence, as before

$$I - M\Delta \text{ is invertible} \qquad \Longleftrightarrow \qquad I - \Theta M \Theta^{-1}\Delta \text{ is invertible}$$

Hence, for any $\Theta \in \mathbf{\Theta}$,

$$\mu(M, \mathbf{\Delta}) = \mu(\Theta M \Theta^{-1}, \mathbf{\Delta})$$

An upper bound for μ

$$\begin{aligned} \mu(M, \mathbf{\Delta}) &\leq \inf\Big\{ \mu(\Theta M \Theta^{-1}, \mathbf{\Delta}) \ ; \ \Theta \in \mathbf{\Theta} \Big\} \\ &\leq \inf\Big\{ \|\Theta M \Theta^{-1}\| \ ; \ \Theta \in \mathbf{\Theta} \Big\} \end{aligned}$$

Hence $\mu(M, \mathbf{\Delta}) < 1$ if

$$\text{there exists } \Theta \in \mathbf{P\Theta} \text{ such that } M^*\Theta M - \Theta < 0$$

The matrix structured singular value

Let $M \in \mathbb{C}^{m \times m}$, and $\mathbf{\Delta} \subseteq \mathbb{C}^{m \times m}$. Also

$$\mathbf{C\Delta}_{\mathsf{s,f}} = \left\{ \Delta = \operatorname{diag}\big(\delta_1 I_{m_1}, \dots, \delta_s I_{m_s}, \Delta_{s+1}, \dots, \Delta_{s+f}\big) \; ; \; \delta_i \in \mathbb{C}, \; \Delta_k \in \mathbb{C}^{m_k \times m_k} \right\}$$

$$\mathbf{\Delta}_{\mathsf{s,f}} = \left\{ \Delta \in \mathbf{C\Delta}_{\mathsf{s,f}} \; ; \; \|\Delta\| \le 1 \right\}$$

$$\mathbf{\Theta}_{\mathsf{s,f}} = \left\{ \operatorname{diag}\big(\Theta_1, \dots, \Theta_s, \theta_{s+1} I, \dots, \theta_{s+f} I\big) \; ; \; \Theta_k \in \mathbb{C}^{m_k \times m_k}, \; \theta_k \in \mathbb{C} \right\}$$

Bounds on μ

$\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

there does not exist $\Delta \in \mathbf{\Delta}_{\mathsf{s,f}}$ such that $I - M\Delta$ is singular

which holds if and only if

there does not exist $\Delta \in \mathbf{\Delta}_{\mathsf{s,f}}$ and $q \in \mathbb{C}^n$ such that $M\Delta q = q$

The matrix structured singular value

$$\mathbf{C\Delta}_{\mathsf{s,f}} = \left\{ \Delta = \operatorname{diag}\big(\delta_1 I_{m_1}, \dots, \delta_s I_{m_s}, \Delta_{s+1}, \dots, \Delta_{s+f}\big) \ ; \ \delta_i \in \mathbb{C}, \ \Delta_k \in \mathbb{C}^{m_k \times m_k} \right\}$$

$\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

there does not exist $\Delta \in \mathbf{\Delta}_{\mathsf{s,f}}$ and $q \in \mathbb{C}^n$ such that $M\Delta q = q$

which holds if and only if

there does not exist $\Delta \in \mathbf{\Delta}_{\mathsf{s,f}}$ and $p, q \in \mathbb{C}^n$ such that
$$\begin{aligned} p &= Mq \\ q &= \Delta p \end{aligned}$$

Partition

$$\Delta = \begin{bmatrix} \delta_1 I & & & & & \\ & \ddots & & & & \\ & & \delta_s & & & \\ & & & \Delta_{s+1} & & \\ & & & & \ddots & \\ & & & & & \Delta_{s+f} \end{bmatrix} \quad M = \begin{bmatrix} M_1 \\ \vdots \\ M_s \\ M_{s+1} \\ \vdots \\ M_{s+f} \end{bmatrix} \quad q = \begin{bmatrix} q_1 \\ \vdots \\ q_s \\ q_{s+1} \\ \vdots \\ q_{s+f} \end{bmatrix} \quad p = \begin{bmatrix} p_1 \\ \vdots \\ p_s \\ p_{s+1} \\ \vdots \\ p_{s+f} \end{bmatrix}$$

The matrix structured singular value

Now $q = \Delta p$ if and only if

$$q_k = \delta_k p_k \text{ for } k = 1, \ldots, s$$
$$q_k = \Delta_k p_k \text{ for } k = s+1, \ldots, s+f$$

Theorem

Given $q, p \in \mathbb{C}^m$, there exists a matrix $\Delta \in \mathbb{C}^{m \times m}$ with $\|\Delta\| \le 1$ such that

$$q = \Delta p$$

if and only if $p^* p - q^* q \ge 0$.

Proof

- *only if:* easy.
- *if:* Choose $\Delta = \dfrac{qp^*}{p^* p}$. Then

$$\|\Delta\|^2 = \|\Delta\Delta^*\| = \frac{\|qq^*\|}{\|p^* p\|} = \frac{\|q^* q\|}{\|p^* p\|}$$

Theorem

Given $q, p \in \mathbb{C}^m$, there exists $\delta \in \mathbb{C}$ with $|\delta| \leq 1$ such that

$$q = \delta p$$

if and only if $pp^* - qq^* \geq 0$.

Proof

- *only if:* $pp^* - qq^* = pp^*(1 - |\delta|^2) \geq 0$
- *if:* If $pp^* - qq^* \geq 0$, then $\ker(p^*) \subset \ker(q^*)$.

  Hence $\operatorname{image}(p) \supset \operatorname{image}(q)$.

  Hence there exists $\delta \in \mathbb{C}$ such that

  $$q = \delta p$$

  Then since $pp^* - qq^* \geq 0$, we have $pp^*(1 - |\delta|^2) \geq 0$ which implies $|\delta| \leq 1$.

The matrix structured singular value

We have $\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

there does not exist $\Delta \in \mathbf{\Delta}_{\mathsf{s,f}}$ and $p, q \in \mathbb{C}^n$ such that

$$q_k = \delta_k p_k \text{ for } k = 1, \dots, s$$
$$q_k = \Delta_k p_k \text{ for } k = s+1, \dots, s+f$$

Hence $\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

there does not exist $p, q \in \mathbb{C}^n$ such that $p_k = M_k q_k$ and

$$p_k p_k^* - q_k q_k^* \geq 0 \qquad \text{for } k = 1, \dots, s$$
$$p_k^* p_k - q_k^* q_k \geq 0 \qquad \text{for } k = s+1, \dots, s+f$$

The matrix structured singular value

We have $\mu(M, \mathbf{\Delta}_{\text{s,f}}) < 1$ if and only if

there does not exist $q \in \mathbb{C}^n$ such that

$$
\begin{aligned}
M_k q q^* M_k^* - q_k q_k^* &\geq 0 \qquad \text{for } k = 1, \ldots, s \\
q^* M_k^* M_k q - q_k^* q_k &\geq 0 \qquad \text{for } k = s+1, \ldots, s+f
\end{aligned}
$$

Define the functions

$$
\begin{aligned}
\Phi_k(q) &= M_k q q^* M_k^* - q_k q_k^* \\
\phi_k(q) &= q^* M_k^* M_k q - q_k^* q_k
\end{aligned}
$$

and

$$\mathbf{\Phi}(q) = \big(\Phi_1(q), \ldots, \Phi_s(q), \phi_{s+1}(q), \ldots, \phi_{s+f}(q)\big)$$

Then $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{V}$ where

$$\mathbb{V} = \mathbb{H}^{m_1} \times \cdots \times \mathbb{H}^{m_s} \times \mathbb{R}^f$$

The matrix structured singular value

We have $\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

$$\text{there does not exist } q \in \mathbb{C}^n \text{ such that } \mathbf{\Phi}(q) \geq 0$$

Notes

- The *positive cone* in $\mathbb{V}$ is

$$\Pi_{\mathsf{s,f}} = \Big\{ Y \in \mathbb{V} \;;\; Y \geq 0 \Big\}$$
$$= \left\{ \begin{aligned} Y = \big(R_1, \dots, R_s, r_{s+1}, \dots, r_{s+f}\big) \;;\; R_k \geq 0 \text{ for } k = 1, \dots, s \\ r_k \geq 0 \text{ for } k = s+1, \dots, s+f \end{aligned} \right\}$$

- Let $\nabla_{\mathsf{s,f}} = \mathrm{image}(\mathbf{\Phi})$.

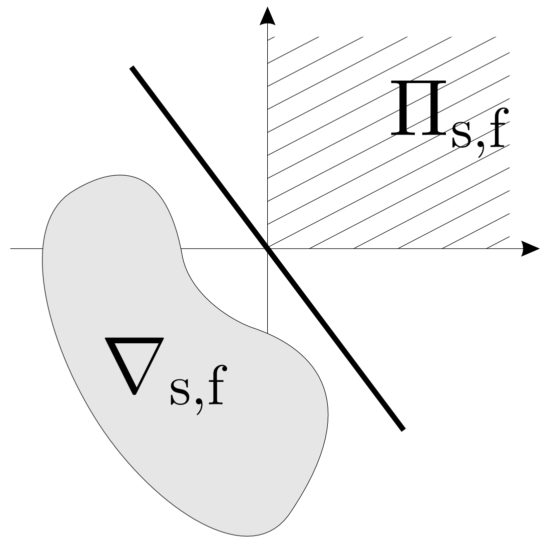
- Then $\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

$$\nabla_{\mathsf{s,f}} \cap \Pi_{\mathsf{s,f}} = \emptyset$$

The matrix structured singular value

$\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

$$\nabla_{\mathsf{s,f}} \cap \Pi_{\mathsf{s,f}} = \emptyset$$



How to test if two sets are disjoint? A sufficient condition is that there exists a *separating hyperplane*. If the two sets are convex, this test is also necessary.

The matrix structured singular value

Recall

$$\mathbb{V} = \mathbb{H}^{m_1} \times \cdots \times \mathbb{H}^{m_s} \times \mathbb{R}^f$$

and the *positive cone* in $\mathbb{V}$ is

$$\begin{aligned} \Pi_{\mathsf{s,f}} &= \Big\{ Y \in \mathbb{V} \; ; \; Y \geq 0 \Big\} \\ &= \left\{ \begin{aligned} Y = \big(R_1, \ldots, R_s, r_{s+1}, \ldots, r_{s+f}\big) \; ; \; R_k \geq 0 \text{ for } k = 1, \ldots, s \\ r_k \geq 0 \text{ for } k = s+1, \ldots, s+f \end{aligned} \right\} \end{aligned}$$

The inner product in $\mathbb{V}$ is

$$\langle Y, R \rangle = \sum_{k=1}^{s} \mathrm{Trace}(Y_k R_K) + \sum_{k=s+1}^{s+f} y_k r_k$$

The matrix structured singular value

We have $\mu(M, \mathbf{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

$$\nabla_{\mathsf{s,f}} \cap \Pi_{\mathsf{s,f}} = \emptyset$$

The positive cone satisfies

$$Y \in \Pi_{\mathsf{s,f}} \qquad \Longleftrightarrow \qquad \langle Y, R \rangle \geq 0 \qquad \text{for all } R \in \Pi_{\mathsf{s,f}}$$

Let $\bar{\Theta} = (\Theta_1, \ldots, \Theta_s, \theta_{s+1}, \ldots, \theta_{s+f})$. Then a separating hyperplane, defined by $\bar{\Theta}$, exists if and only if

$$\langle \bar{\Theta}, Y \rangle < 0 \qquad \text{for all } Y \in \nabla_{\mathsf{s,f}}$$

This condition is

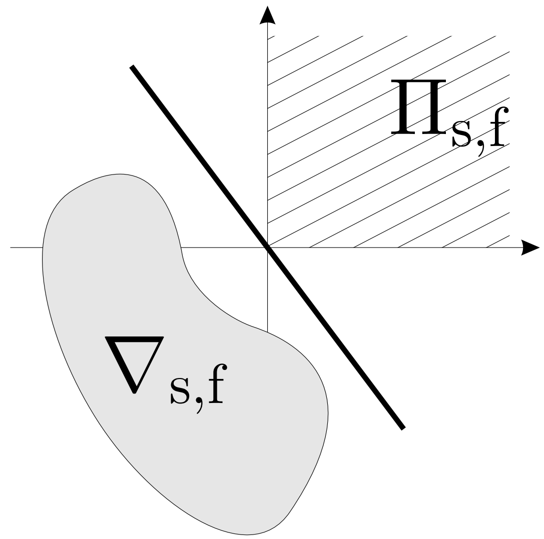
$$\sum_{k=1}^{s} \operatorname{Trace}\bigl(\Theta_k (M_k q q^* M_k^* - q_k q_k^*)\bigr) + \sum_{k=s+1}^{s+f} \theta_k (q^* M_k^* M_k q - q_k^* q_k) < 0 \text{ for all } \Theta \in \nabla_{\mathsf{s,f}}$$

Rearrangement of this inequality shows that it is equivalent to

$$M^* \Theta M - \Theta < 0$$

for $\Theta = \operatorname{diag}(\Theta_1, \ldots, \Theta_s, \theta_{s+1}, \ldots, \theta_{s+f})$.

Summary



- $\mu(M, \boldsymbol{\Delta}_{\mathsf{s,f}}) < 1$ if and only if

$$\nabla_{\mathsf{s,f}} \cap \Pi_{\mathsf{s,f}} = \emptyset$$

- There exists a separating hyperplane if and only if there exists $\Theta \in \boldsymbol{\Theta}_{\mathsf{s,f}}$ such that

$$M^*\Theta M - \Theta < 0$$

- Hence this condition is necessary and sufficient for robust well-connectedness when $\nabla_{\mathsf{s,f}}$ is convex.
- Otherwise, it is only a sufficient condition.