Engr210a Lecture 2: Mathematical Preliminaries

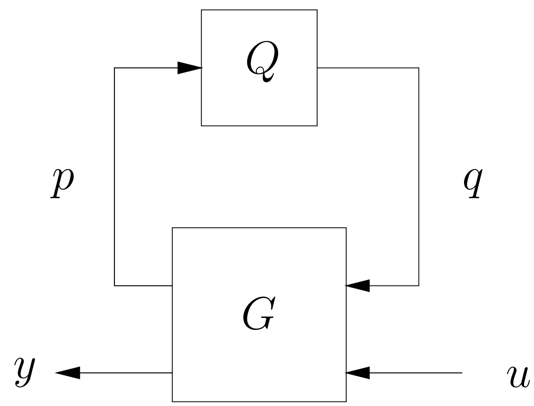
- Recap intro on uncertain systems
- Subspaces, affine sets, convex sets, convex cones
- Hyperplanes and separating hyperplanes
- Matrix inequalities
- Schur complements

Representing nonlinear systems

We can write any nonlinear system

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = 0$$
$$y(t) = g(x(t), u(t))$$

as a linear system G connected to a nonlinear system Q.

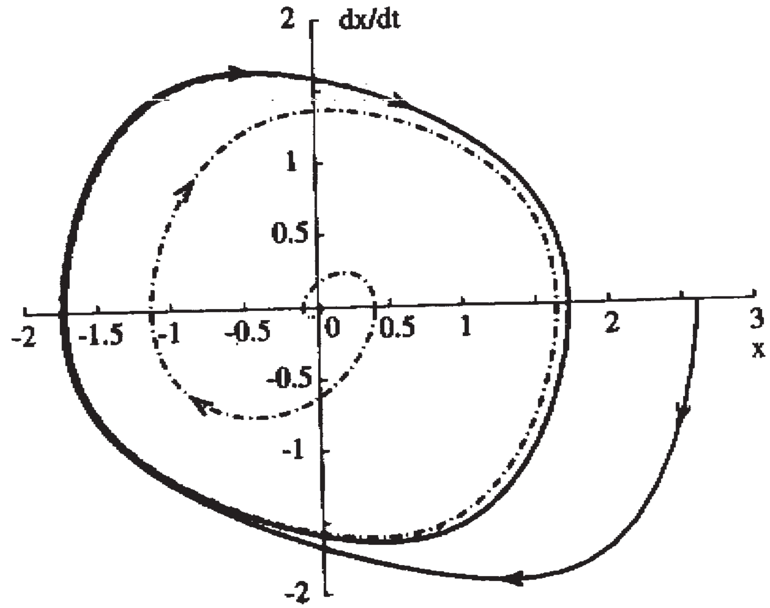


We usually choose coordinates so that

$$f(0,0) = 0 \qquad g(0,0) = 0$$

Example: Raleigh equation

$$\ddot{y} - 2\zeta(1 - \alpha\dot{y}^2)\dot{y} + y = u$$



Define G by

$$\dot{x}(t) = \begin{bmatrix} 2\zeta & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} -2\zeta\alpha \\ 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$p(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

and Q by

$$q(t) = Q(p(t)) = p(t)^3$$

Replacing or Modeling Q

$$\begin{aligned}\mathcal{Q} &= \Big\{(p,q) : q = Q(p)\Big\} \\ &= \Big\{\text{set of input-output signal pairs}\Big\}\end{aligned}$$

Consider a set of maps

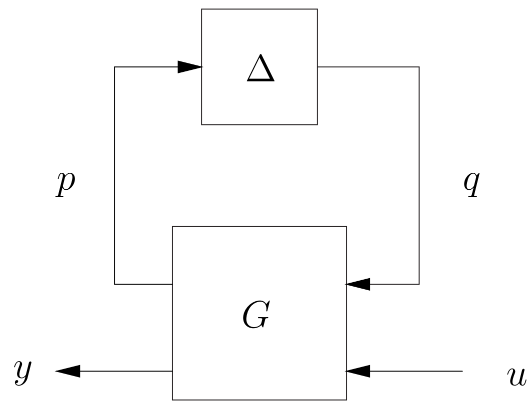
$$\boldsymbol{\Delta} = \Big\{\Delta : \{\text{input-signals } p\} \to \{\text{output signals } q\}\Big\}$$

with the following property:

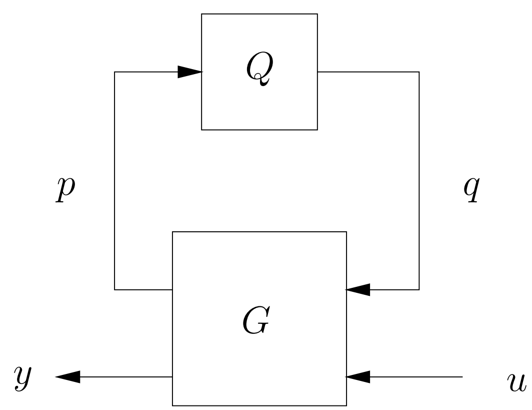
if $(p,q) \in \mathcal{Q}$ then there exists $\Delta \in \boldsymbol{\Delta}$ such that $q = \Delta(p)$.

Using uncertain systems

If a property holds for each of the systems



then it holds for the original system



Subspaces

Suppose V is a vector space. Then $S \subseteq V$ is a *subspace* if

$$x, y \in S \qquad \lambda, \mu \in \mathbb{R} \qquad \Longrightarrow \qquad \lambda x + \mu y \in S$$

Geometrically: if $x, y \in S$ then the plane through $x, y, 0$ is contained in S.

Representations

As the image of a linear operator $A : U \to V$.

$$\begin{aligned} S &= \operatorname{image}(A) \\ &= \operatorname{range}(A) \end{aligned}$$

Recall we say A is *surjective* if $\mathrm{image}(A) = V$.

One may also represent a subspace as the span of a set of vectors

$$\begin{aligned} S &= \operatorname{span}\{a_1, a_2, \ldots, a_n\} \\ &= \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \ ; \ \lambda_i \in \mathbb{R}\} \end{aligned}$$

In finite dimensions, A has a matrix representation $A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$

Subspace representations. . .

We can also represent $S \subseteq V$ as the kernel of a linear operator $B : V \to U$.

$$\begin{aligned} S &= \operatorname{nullspace}(B) \\ &= \ker(B) \\ &= \{x \in V \ ; \ Bx = 0\} \end{aligned}$$

Or as the set of vectors orthogonal to another set

$$S = \{x \in V \ ; \ \langle b_1, x\rangle = 0, \langle b_2, x\rangle = 0, \ldots, \langle b_n, x\rangle = 0\}$$

Recall we say B is *injective* if $\ker(B) = \{0\}$.

In finite dimensions, B has a matrix representation

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

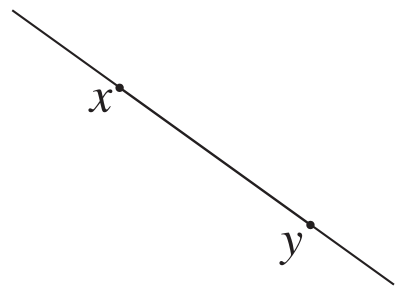
and often in real Euclidean space $\langle b_i, x\rangle = b_i^T x$.

Affine Sets

Suppose V is a vector space. Then $S \subseteq V$ is an *affine set* if

$$x, y \in S \qquad \lambda, \mu \in \mathbb{R}, \qquad \lambda + \mu = 1 \implies \qquad \lambda x + \mu y \in S$$

Geometrically: if $x, y \in S$ then the line through x, y is contained in S.



Representations

As the range of the affine function $Au + b$, where $A : U \to V$

$$S = \{Au + b \; ; \; u \in U\}$$

As the solution to linear equations, where $B : V \to U$

$$S = \{x \; ; \; Bx = c\}$$

Or, we can represent the linear equations as

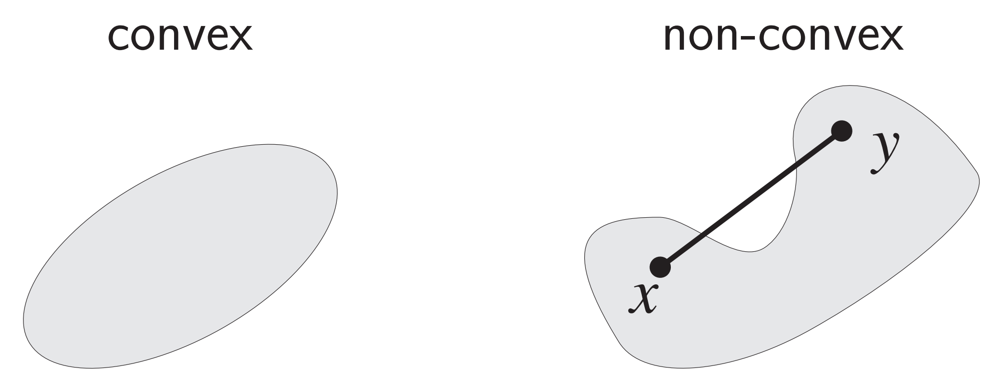
$$S = \{x \; ; \langle b_1, x\rangle = c_1, \langle b_2, x\rangle = c_2, \dots, \langle b_n, x\rangle = c_n\}$$

Convex Sets

The set $S \subseteq V$ is a *convex set* if

$$x, y \in S \qquad \lambda, \mu \geq 0, \qquad \lambda + \mu = 1 \qquad \Longrightarrow \qquad \lambda x + \mu y \in S$$

Geometrically: S is convex if, given $x, y \in S$, the line segment $\mathcal{L}(x, y) \subseteq S$



The line segment

$$\begin{aligned} \mathcal{L}(x, y) &= \{v \in V \; ; \; v = \lambda x + \mu y \text{ for some } \lambda, \mu \geq 0, \lambda + \mu = 1\} \\ &= \{v \in V \; ; \; v = \theta x + (1 - \theta) y \text{ for some } \theta \in [0, 1]\} \end{aligned}$$

Representations

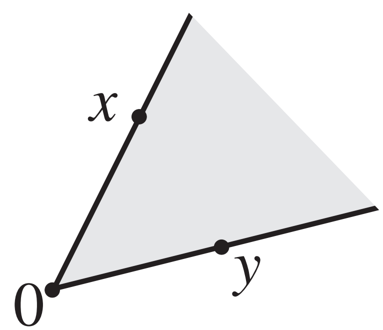
Ellipsoids, polyhedra, and many others. We will see more later in the course.

Convex Cones

The set $S \subseteq V$ is a *convex cone* if

$$x, y \in S \qquad \lambda, \mu \geq 0, \qquad \Longrightarrow \qquad \lambda x + \mu y \in S$$

Geometrically: S is convex if, given $x, y \in S$, the 'pie slice' between x and Y is contained in S.



Representations

Many representations . . .

Hyperplane

In an n dimensional vector space, an $n-1$ dimensional affine set is called a *hyperplane*.

Representations

A linear mapping $F : V \to \mathbb{R}$ is called a *linear functional*.

Given F, a hyperplane can be represented as the set of $x \in V$ which satisfy

$$S = \{x \; ; \; Fx = c\}$$

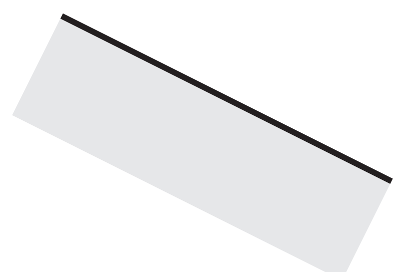
Or, given a dual vector b,

$$S = \{x \; ; \langle b, x\rangle = c\}$$

Think of b as the normal to S.

Halfspace

$$S = \{x \; ; \; Fx \le c\} \qquad \text{or} \qquad S = \{x \; ; \langle b, x\rangle \le c\}$$



Intersections

If S_i is a subspace for all $i \in \mathcal{I}$, then

$$\bigcap_{i \in \mathcal{I}} S_i \text{ is a subspace.}$$

Similarly for affine sets, convex sets and convex cones.

Fact: Every closed convex set is the intersection of a (usually infinite) set of halfspaces.

In fact, if S is a closed convex set then

$$S = \bigcap \Big\{ H \; ; \; H \text{ is a closed halfspace, and } S \subseteq H \Big\}$$

Example: A polyhedron is the intersection of finitely many halfspaces.

Convex Hull

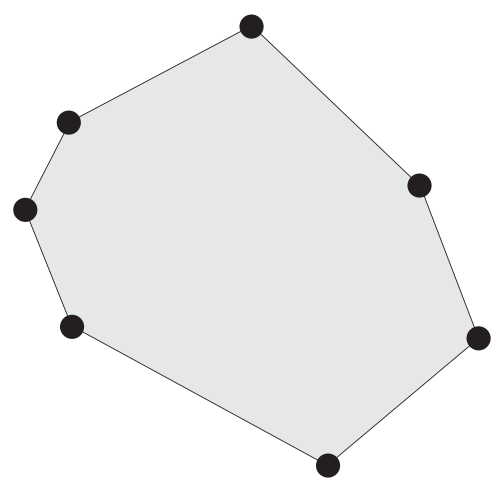
Given a subset $S \subseteq V$, the *convex hull* of S is the smallest convex set containing S.

$$\mathrm{Co}(S) = \bigcap \Big\{ C \subseteq V, C \text{ is convex}, S \subseteq C \Big\}$$

Given $S = \{x_1, x_2, \ldots, x_n\}$,

$$\mathrm{Co}(S) = \Big\{ \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n \ ; \ \sum_{i=1}^{n} \theta_1 = 1, \theta_i \geq 0 \Big\}$$

Example:



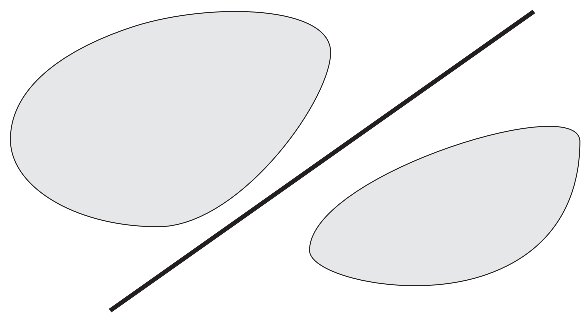
Separating hyperplanes

Every hyperplane $H \subset V$ always breaks V into two half-spaces, which have the form

$$\{v\ ;\ F(v) \leq a\} \qquad \text{and} \qquad \{v\ ;\ F(v) \geq a\}$$

Given two sets $S, T \subset V$, we say that the hyperplane separates S and T if

$$S \subseteq \{v\ ;\ F(v) \leq a\} \qquad \text{and} \qquad T \subseteq \{v\ ;\ F(v) \geq a\}$$



Separating Hyperplane Theorem:

If S and T are nonempty, convex and $S \cap T = \emptyset$ then there exists a hyperplane which separates S from T.

Stronger form: There exists a strictly separating hyperplane if and only if the sets are strictly separated.

The vector space of matrices

Linear functionals: Given $Y \in \mathbb{R}^{n \times n}$

$$F(X) = \operatorname{trace}(Y^* X) = \sum_{i,k=1}^{n} x_{ik} y_{ik}$$

is a linear functional on $\mathbb{R}^{n \times n}$. Every linear functional on $\mathbb{R}^{n \times n}$ can be represented this way.

Symmetric matrices

The set of symmetric matrices

$$\mathbb{S}^n = \{X \in \mathbb{R}^{n \times n} \; ; \; X = X^T\}$$

is a subspace of $\mathbb{R}^{n \times n}$.

Similarly, every linear functional on $\mathbb{S}^n$ is represented by

$$F(X) = \operatorname{trace}(Y^* X)$$

for some $Y \in \mathbb{S}^n$.

Eigenvectors of symmetric matrices

Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric, i.e. $A = A^T$.

Theorem: The eigenvalues of A are real.
Proof: Suppose $Ax = \lambda x$ for $x \neq 0$. Then

$$\lambda x^* x = x^* A x = (Ax)^* x = \lambda^* x^* x$$

Since $x^* x > 0$ we can conclude that $\lambda = \lambda^*$.

Theorem: There is a set $\{q_1, q_2, \ldots, q_n\}$ of n orthonormal eigenvectors of A, which satisfy

$$A q_i = \lambda_i q_i, \qquad q_i^* q_j = \delta_{ij}$$

In matrix form, there is an orthogonal matrix U such that

$$U^{-1} A U = U^* A U = \Lambda$$

hence we can express A as

$$A = U \Lambda U^* = \sum_{i=1}^{n} \lambda_i q_i q_i^*$$

Matrix Inequalities

The set of Hermitian matrices

$$\mathbb{H}^n = \{A \in \mathbb{R}^{n\times n} \; ; \; A = A^*\}$$

is a subspace of $\mathbb{R}^{n\times n}$.

We say A is *positive semidefinite* if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$.

- Write this as $A \geq 0$.
- $A \geq 0$ if and only if $\lambda_i(A) \geq 0$ for all i.

We say A is *positive definite* if $x^*Ax > 0$ for all $x \in \mathbb{C}^n$.

- Write this as $A > 0$.
- $A > 0$ if and only if $\lambda_i(A) > 0$ for all i.

We say A is *negative definite*, written $A < 0$, if $-A > 0$.

Similarly for *negative semidefinite*.

We say $A > B$ if $A - B > 0$.

Otherwise A is called *indefinite*.

Positive definiteness

- If $Q > 0$ and $A \in \mathbb{C}^{n \times m}$, then $A^*QA \geq 0$. If $\ker(A) = 0$, then $A^*QA > 0$.
- If $Q_1, Q_2 \geq 0$ and $\mu_1, \mu_2 \geq 0$, then $\mu_1 Q_1 + \mu_2 Q_2 \geq 0$. This implies that the set of positive definite matrices is a *convex cone*.

The Schur complement formula

Suppose Q, M and R are matrices and that M and Q are Hermitian. Then the following are equivalent:

(a) The matrix inequalities $Q > 0$ and

$$M - RQ^{-1}R^* > 0 \text{ both hold.}$$

(b) The matrix inequality

$$\begin{bmatrix} M & R \\ R^* & Q \end{bmatrix} > 0 \text{ is satisfied.}$$

Proof of the Schur complement formula

The two inequalities listed in (a) are equivalent to the single block inequality.

$$\begin{bmatrix} M - RQ^{-1}R^* & 0 \\ 0 & Q \end{bmatrix} > 0 \ .$$

Now left- and right-multiply this inequality by the nonsingular matrix

$$\begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix}$$

and its adjoint, respectively, to get

$$\begin{bmatrix} M & R \\ R^* & Q \end{bmatrix} = \begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} M - RQ^{-1}R^* & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ Q^{-1}R^* & I \end{bmatrix} > 0.$$

Therefore inequality (b) holds if and only if (a) holds.