Engr210a Lecture 5: Linear analysis

- Norms on vectors, signals, and matrices

- Inner products

- Topology

- Convergence and completeness

- Bounded operators

- Induced norms

Norms

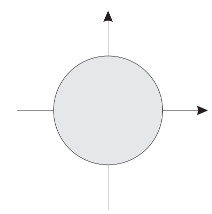
A *norm* $\|\cdot\|$ on a vector space $\mathcal{V}$ is a function mapping $\mathcal{V} \to [0, \infty)$ which satisfies

- *Definiteness:* $\|v\| = 0$ if and only if $v = 0$
- *Homogeneity:* $\|\alpha v\| = |\alpha|\|v\|$
- *Triangle inequality:* $\|u + v\| \leq \|u\| + \|v\|$

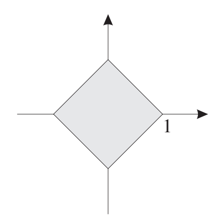
for all $u, v \in \mathcal{V}$ and $\alpha \in \mathbb{R}$.

Examples on $\mathbb{C}^n$

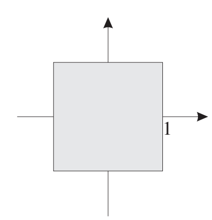
- *Euclidean norm:* $\|v\|_2 = v^* v = \left(|v_1|^2 + \cdots + |v_n|^2\right)^{\frac{1}{2}}$



- *p-norm:* $\|v\|_p = \left(|v_1|^p + \cdots + |v_n|^p\right)^{\frac{1}{p}}$
- *1-norm:* $\|v\|_1 = |v_1| + \cdots + |v_n|$



- ∞-norm: $\|v\|_\infty = \max\left\{|v_k| \; ; \; k = 1, \ldots, n\right\}$



Norms on matrices

For a matrix $A \in \mathbb{C}^{m \times n}$, we define

- *Frobenius norm:* $\|A\|_F = \big(\operatorname{trace}(A^*A)\big)^{\frac{1}{2}}$

  If $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, then

$$\begin{aligned} \|A\|_F &= \left(\|a_1\|_2^2 + \|a_2\|_2^2 + \cdots + \|a_n\|_2^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|A_{ij}|^2\right)^{\frac{1}{2}} \end{aligned}$$

- *Spectral norm:*

$$\begin{aligned} \|A\| &= \max\left\{\lambda^{\frac{1}{2}} \ ; \ \lambda \text{ is an eigenvalue of } A^*A\right\} \\ &= \max\left\{\lambda^{\frac{1}{2}} \ ; \ \lambda \text{ is an eigenvalue of } AA^*\right\} \\ &= \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \end{aligned}$$

Norms on signals

The L_∞ space of signals on $(-\infty, \infty)$ is

$$L_\infty(-\infty,\infty) = \left\{ u : \mathbb{R} \to \mathbb{C}^n \ ; \ \|u\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|u(t)\|_\infty \text{ is finite} \right\}$$

Notes

- $\operatorname{ess\,sup}_{t\in\mathbb{R}} \|u(t)\|_\infty < 1$ means $\|u(t)\|_\infty < 1$ at all t except a finite set of points $\{t_i\}$.
- Corresponds to the *peak* of a signal.
- The *spatial* norm is also the ∞-norm.

The L_2 space of signals on $(-\infty, \infty)$ is

$$L_2(-\infty,\infty) = \left\{ u : \mathbb{R} \to \mathbb{C}^n \ ; \ \|u\|_2 = \left(\int_{-\infty}^{\infty} \|u(t)\|_2^2 \, dt \right)^{\frac{1}{2}} \text{ is finite} \right\}$$

Notes

- $u(t)$ must decay to zero as $t \to \infty$ for $\|u\|_2$ to be finite.
- Corresponds to the *total energy* in a signal.

Inner product spaces

An inner product $\langle \cdot, \cdot \rangle$ on a vector space $\mathcal{V}$ is a function mapping $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$ which satisfies

- $\langle v, v \rangle \geq 0$, with $\langle v, v \rangle = 0$ if and only if $v = 0$
- $\langle v, \alpha_1 u_1 + \alpha_2 u_2 \rangle = \alpha_1 \langle v, u_1 \rangle + \alpha_2 \langle v, u_2 \rangle$
- $\langle u, v \rangle$ is the complex conjugate of $\langle v, u \rangle$

for all $u_1, u_2, u, v \in \mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Standard facts

- The inner-product captures the idea of *angle* between two vectors. If $\langle u, v \rangle = 0$ we say u, v are *orthogonal*.
- We can define $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$. This is a norm; i.e. it satisfies the required properties of a norm.
- *The Cauchy-Schwartz Inequality:*

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

Examples of inner product spaces

- Euclidean space: $\langle x, y \rangle = x^* y$. The corresponding norm is the 2-norm.
- Matrices $C^{m \times n}$: $\langle A, B \rangle = \mathrm{Trace}(A^* B)$. The corresponding norm is the Frobenius norm.

The space L_2

- The inner product on $L_2(-\infty, \infty)$ is

$$\langle x, y \rangle = \int_{-\infty}^{\infty} \big(x(t)\big)^* y(t)\, dt$$

- $L_2[0, \infty)$ is a subspace of $L_2(-\infty, \infty)$.

The open ball

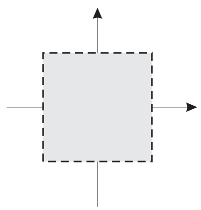
Suppose V is a normed space. The *open ball* $B(x, \varepsilon) \subset V$ is the set

$$B(x, \varepsilon) = \Big\{ y \in \mathcal{V} \; ; \; \|x - y\| < \varepsilon \Big\}$$

We say the ball has *radius* ε.

Example: In $\mathbb{R}^2$, with the ∞-norm, the ball of radius one at the origin is the square

$$B(0, 1) = \Big\{ (x_1, x_2) \in \mathbb{R}^2 \quad |x_1| < 1 \text{ and } |x_2| < 1 \Big\}$$



Note that the boundary is not included.

Interior points

Suppose $P \subseteq V$. Then the point $x \in P$ is an *interior point* if

$$\text{there exists } \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subset P$$

Examples

- If P is the usual unit disk in $\mathbb{R}^2$ including boundary

  $$\left\{ (x_1, x_2) \in \mathbb{R}^2 \quad x_1^2 + x_2^2 \leq 1 \right\}$$

  Then the interior points are those points x which satisfy $x_1^2 + x_2^2 < 1$.

- If P is the plane $x_3 = 0$ in $\mathbb{R}^3$, then P has no interior points.

Notes

- In finite dimensional space, the set of interior points of a set P is the same no matter which norm is used in the definition of the ball.

- In infinite dimensions, this is not true in general.

Open sets

The subset $P \subseteq V$ is called *open* if every point $x \in P$ is an interior point.

Examples

- The *open ball* $B(x, \varepsilon)$ is open for all $x \in V$ and all $\varepsilon > 0$.
- The interval $(a, b) \subset \mathbb{R} = \{x \in \mathbb{R} \; ; \; a < x < b\}$ is open.
- The subset of $\mathbb{R}^2$ defined by

$$\{(x_1, x_2) \; ; \; a < x_1 < b; x_2 = 0\}$$

  is *not open*.

Closure points

A point $x \in V$ is called a *closure point* of the subset $P \subseteq V$ if

$$B(x, \varepsilon) \cap P \neq \emptyset \text{ for all } \varepsilon > 0.$$

That is, x is a closure point of P if every open ball around x intersects with P.

Example of closure points

- The closure points of the half-space in $\mathbb{R}^2$

$$\{(x_1, x_2)\ ;\ x_1 < b\}$$

is the half-space

$$\{(x_1, x_2)\ ;\ x_1 \leq b\}$$

Closed sets

The subset $P \subseteq V$ is called *closed* if every closure point x of P is an element of P.

Facts

- The complement of an open set is closed and the complement of a closed set is open.
- The whole space $\mathcal{V}$ is both closed and open.
- If P_1 and P_2 are open, then both $P_1 \cap P_2$ and $P_1 \cup P_2$ are open.
- If P_1 and P_2 are closed, then both $P_1 \cap P_2$ and $P_1 \cup P_2$ are closed.

Caveat:

The intersection of an infinite number of open sets may not be open; for example, $\bigcap\{(1/k, 1/k) \subset \mathbb{R}\ ;\ k = 1, 2, \dots\} = \{0\}$ is not open.

Convergence

The sequence of elements $\{v_0, v_1, v_2, \ldots, \; ; \; v_i \in \mathcal{V}\}$ in a normed space $\mathcal{V}$ *converges* if there exists an element $x \in \mathcal{V}$ such that limit

$$\lim_{k \to \infty} \|v_k - x\| = 0$$

Note that

- Convergence in $\mathcal{V}$ is defined in terms of convergence of real numbers.
- Whether a sequence converges or not depends on which norm is used.
- We need to know x to apply this definition.

Cauchy sequences

The sequence $\{v_0, v_1, v_2, \ldots,\}$ is *Cauchy* if for any $\varepsilon > 0$,

there exists N such that $\quad \|v_i - v_j\| < \varepsilon$ for all $i, j > N$

Notes

- A Cauchy sequence is one that *appears to converge*.

Facts

- Every convergent sequence is Cauchy.
- In a closed subset P of a normed space, every Cauchy sequence converges.

Completeness

The space $\mathcal{V}$ is called *complete* if every Cauchy sequence converges.

- A complete normed space is called a *Banach space*.
- A complete inner-product space is called a *Hilbert space*.

Example

The inner-product space

$$\mathcal{W} = \Big\{ w \in L_2[0,\infty) \;;\; \text{ there exists } T > 0 \text{ such that } w(t) = 0 \text{ for all } t \geq T \Big\}$$

is not complete.

To see this, consider the sequence $\{w_0, w_1, \dots\} \subset \mathcal{W}$

$$w_k(t) = \begin{cases} e^{-t} & \text{for } 0 \leq t \leq k \\ 0 & \text{otherwise} \end{cases}$$

This sequence is Cauchy, since, for $l < k$

$$\begin{aligned} \|w_k - w_l\|^2 &= \int_l^k e^{-2t}\, dt \\ &= \frac{1}{2}(e^{-2l} - e^{-2k}) \end{aligned}$$

with a similar result for $k < l$, and hence $\|w_k - w_l\| \leq e^{-\min\{k,l\}}$.

The sequence $\{w_0, w_1, \dots\} \subset L_2[0,\infty)$, and in that space it converges to the function $w(t) = e^{-t}$. But $w \notin \mathcal{W}$ and hence $\mathcal{W}$ is not complete.

Operators

Linearity

The mapping $F : \mathcal{V} \to \mathcal{Z}$ is called *linear* if

$$F(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 F(v_1) + \alpha_2 F(v_2)$$

for all $v_1, v_2 \in \mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Boundedness

The mapping $F : \mathcal{V} \to \mathcal{Z}$ is called *bounded* if there exists $K > 0$ such that

$$\|F(v)\| \leq K\|v\|$$

for all $v \in \mathcal{V}$.

- A bounded linear map is often called a linear *operator*.
- If F is linear, we often omit the brackets around its argument, and write $Fv = F(v)$, as in matrix multiplication.

Induced-norms

The *induced-norm* of a linear operator is

$$\|F\| = \sup_{v \neq 0} \frac{\|Fv\|}{\|v\|}$$

This is also often called the *operator norm*.

Notes

- $\|F\|$ measures the *maximum amplification* or *gain* of F.

Examples

- If $M : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, and we put the 2-norm on $\mathbb{R}^m$ and $\mathbb{R}^n$, then

$$\|M\| = \overline{\sigma}(M) \qquad \text{the maximum singular value of } M$$

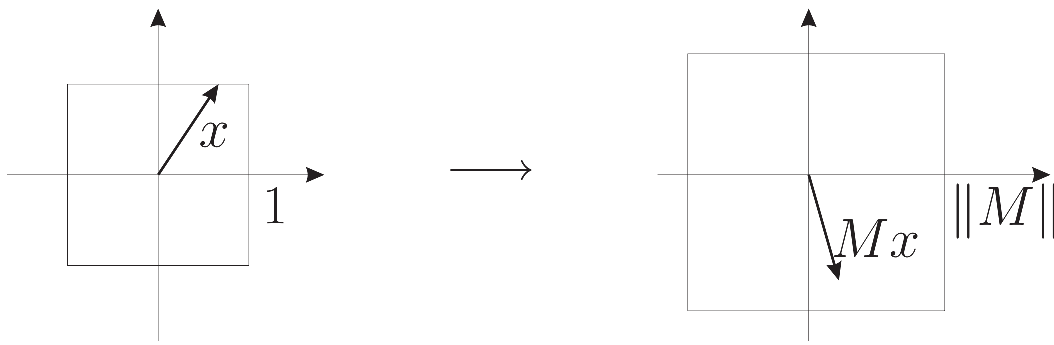
- If we use the ∞-norm on $\mathbb{R}^m$ and $\mathbb{R}^n$, then

$$\|M\|_{\infty \to \infty} = \max_{1 \le i \le m} \sum_{j=1}^{n} |M_{ij}|$$

Interpretation of the induced ∞-norm

The induced ∞-norm of a matrix M is

$$\|M\|_{\infty\to\infty} = \max_{1\le i\le m} \sum_{j=1}^{n} |M_{ij}|$$



Interpretations

- If x fits within a square box of half-width 1, then Mx fits within a square box of half-width $\|M\|_{\infty\to\infty}$.
- The worst-case unit-norm x has the form $x_i = \pm 1$.

Example

$$M = \begin{bmatrix} -1 & 2 & -6 \\ 0 & -2 & 1 \end{bmatrix} \qquad \|M\| = 9 \qquad \text{Worst-case } x = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

The induced ∞-norm of a linear system

Recall the ∞-norm of a signal $w \in L_\infty[0, \infty)$ is

$$\begin{aligned} \|w\|_\infty &= \operatorname*{ess\,sup}_t \ \|w(t)\|_\infty \\ &= \operatorname*{ess\,sup}_t \ \sup_i \ |w_i(t)| \end{aligned}$$

Consider the state-space linear system G given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

with u and y scalar real-valued signals.

Suppose we know the $\|u\|_\infty$; that is, the worst-case peak of u is given by

$$|u(t)| \le u_{\mathsf{peak}}$$

We would like to find the worst-case peak value of the output signal y. This is given by

$$u_{\mathsf{peak}} \|G\|_{\infty \to \infty}$$

How do we compute $\|G\|_{\infty \to \infty}$?

The induced ∞-norm of a linear system

Recall G can be represented as the convolution

$$(Gu)(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$$

where $g(t) = Ce^{At}B$ is the *impulse response* of G.

Theorem

The induced L_∞ norm of G is the 1-norm of the impulse response.

$$\|G\|_{\infty\to\infty} = \|g\|_1 = \int_0^\infty |g(t)|\,dt$$

Proof: We have

$$\begin{aligned}
|y(t)| &= \left| \int_0^t g(t-\tau)u(\tau)\,d\tau \right| \\
&\leq \int_0^t |g(t-\tau)||u(\tau)|\,d\tau \\
&\leq \int_0^\infty |g(t-\tau)|\,d\tau\,\|u\|_\infty = \|g\|_1\,\|u\|_\infty
\end{aligned}$$

Hence we have $\|G\|_{\infty\to\infty} \leq \|g\|_1$.

Proof continued

We have shown that $\|G\|_{\infty\to\infty} \leq \|g\|_1$. We need to show that $\|G\|_{\infty\to\infty} \geq \|g\|_1$. Ideally we would like to find u such that

$$\|y\|_\infty = \|g\|_1 \|u\|_\infty$$

We need only consider u with $\|u\|_\infty = 1$. In fact we will show that for any $\varepsilon > 0$, we can find a u such that

$$\|y\|_\infty \geq \|g\|_1 - \varepsilon$$

Choose t such that

$$\int_0^t |g(t-\tau)|\, d\tau \geq \|g\|_1 - \varepsilon$$

and set $u(\tau) = \operatorname{sgn}(g(t-\tau))$. Then

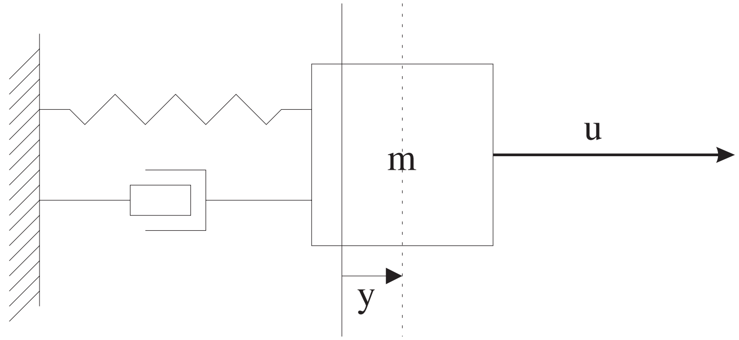
$$y(t) = \int_0^t g(t-\tau)u(\tau)\, d\tau = \int_0^t |g(t-\tau)|\, d\tau$$

Since y is continuous, we have

$$\|y\|_\infty \geq \|g\|_1 - \varepsilon$$

In general, the optimal input on $[0, T]$ is $u(t) = \operatorname{sgn}(g(T-t))$. Note, this is *not* a sinusoid.

Example: mass-spring system



We have, with $m = 1$, $d = 0.2$ and $k = 0.2$,

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

Numerically, we find $\|g\|_1 = 1.447$ and $\|y\|_\infty = 1.430$.