Engr210a Lecture 6: Linear analysis and systems

- Banach algebras
- Invertibility of operators
- The small-gain theorem
- The spectrum of an operator
- Adjoint operators
- Signal spaces L_2 and H_2
- The Fourier and Laplace transforms
- Time-invariance and causality
- Operator spaces L_∞ and H_∞

Operators

For Banach spaces $\mathcal{V}$ and $\mathcal{Z}$, the map $F : \mathcal{V} \to \mathcal{Z}$ is a bounded linear operator if

- *Linearity:* $F(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 F(v_1) + \alpha_2 F(v_2)$ for all $v_1, v_2 \in \mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.
- *Boundedness:* There exists $K > 0$ such that $\|F(v)\| \leq K\|v\|$ for all $v \in \mathcal{V}$.

Sets of linear operators

- $\mathcal{L}(\mathcal{V}, \mathcal{Z})$ is the set of all bounded linear operators mapping $\mathcal{V}$ to $\mathcal{Z}$.
- $\mathcal{L}(\mathcal{V})$ is the set of all bounded linear operators mapping $\mathcal{V}$ to itself.

The set $\mathcal{L}(\mathcal{V}, \mathcal{Z})$ is a *Banach space.*

- It is a vector space; we have addition and scalar multiplication. e.g.

$$(F_1 + F_2)(v) = F_1(v) + F_2(v)$$

- It has a norm – the induced norm.
- It is *complete*. We will not prove this here.

Banach algebras

As well as being a normed vector space, the set $\mathcal{L}(\mathcal{V})$ has additional structure, since one may compose maps. We write $(F_1 F_2)(v) = F_1(F_2(v))$, giving

$$F_1, F_2 \in \mathcal{L}(\mathcal{V}) \qquad \Longrightarrow \qquad F_1 F_2 \in \mathcal{L}(\mathcal{V})$$

The space $\mathcal{L}(\mathcal{V})$ is called a *Banach algebra*

Axiomatic definition of a Banach algebra

- There exists an element $I \in \mathcal{B}$, such that $F \cdot I = I \cdot F = F$, for all $F \in \mathcal{B}$.
- $F(GH) = (FG)H$, for all F, G, H in $\mathcal{B}$.
- $F(G + H) = FG + FH$, for all F, G, H in $\mathcal{B}$.
- For all F, G in $\mathcal{B}$, and each scalar α, we have $F(\alpha G) = (\alpha F)G = \alpha FG$.
- The submultiplicative inequality: $\|FG\| \leq \|F\| \, \|G\|$.

The submultiplicative inequality

The *submultiplicative inequality* is

$$\|FG\| \leq \|F\| \, \|G\|$$

- This is very useful in control; leads to a useful robustness test.
- It follows from the definition of the induced-norm:

$$\|FGx\| \leq \|F\| \, \|Gx\| \leq \|F\| \, \|G\| \, \|x\|$$

Examples

- The set of linear operators on any Banach space $\mathcal{V}$ forms a Banach algebra.
- The set of $n \times n$ matrices forms a Banach algebra.

Invertibility of operators

An operator $F \in \mathcal{L}(\mathcal{V})$ is called *invertible* if there exists $G \in \mathcal{L}(\mathcal{V})$ such that

$$FG = I \text{ and } GF = I$$

We write $G = F^{-1}$ as usual.

Note that the inverse G must be *bounded*, and that we need both equations to hold. For example, ℓ_2 is the space of square-summable sequences

$$\ell_2 = \left\{ (x_0, x_1, \dots) \; ; \; x_i \in \mathbb{R}^n, \; \sum_{i=0}^{\infty} x_i^* x_i \text{ is finite} \right\}$$

Consider the *forward shift operator* $Z \in \mathcal{L}(\ell_2)$ where $y = Zx$ if

$$y_k = \begin{cases} x_{k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}$$

This maps $(10, 3, 2, \dots)$ to $(0, 10, 3, 2, \dots)$.

The *backward shift operator* $B \in \mathcal{L}(\ell_2)$ defined by

$$y = Bx \qquad \text{if } y_k = x_{k+1} \text{ for all } k \geq 0$$

satisfies $BZ = I$, but not $ZB = I$. The operator Z is called *not invertible* or *singular*, even though given y one can find x.

The small-gain theorem

Suppose Q is an element of a Banach algebra $\mathcal{B}$. Then

$$\|Q\| < 1 \qquad \Longrightarrow \qquad I - Q \text{ is invertible, and } (I-Q)^{-1} = \sum_{i=0}^{\infty} Q^k$$

Examples

- If $Q = \begin{bmatrix} 0 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$, then $\|Q\| = \overline{\sigma}(Q) = 0.72$. Then we know that $I - Q$ is invertible.
- Clearly the reverse implication does not hold. For example, $Q = 2I$.

Notes

- Here we are only assuming that Q is an element of a Banach algebra. We do not need to use any properties of Q as a linear map.
- The submultiplicative property implies $\|PQ\| \leq \|P\|\,\|Q\|$. Hence if $\|P\| \leq 1$

  $$I - PQ \text{ is invertible for all operators } Q \text{ with } \|Q\| < 1.$$

  This is very useful when analyzing stability of feedback loops.

Left and right inverses

For $F \in \mathcal{B}$, The operator $L \in \mathcal{B}$ is called the *left-inverse* of F if $LF = I$.

Similarly, $R \in \mathcal{B}$ is called the *right-inverse* of F if $FR = I$.

If F has both a left-inverse and a right-inverse, then they are equal, since

$$L = L(BR) = (LB)R = R$$

Series convergence

The infinite sum is defined by

$$\sum_{i=0}^{\infty} Q^i = \lim_{n \to \infty} T_n$$

where T_n is the *partial sum*

$$T_n = \sum_{i=0}^{n} Q^i$$

Proof of the small-gain theorem

First, we show $\sum_{i=0}^{\infty} Q^i$ is in the Banach algebra $\mathcal{B}$. We need to show that $\{T_0, T_1, \dots\}$ is a Cauchy sequence. For $m > n$,

$$\|T_m - T_n\| = \left\| \sum_{i=n+1}^{m} Q^i \right\| \le \sum_{i=n+1}^{m} \|Q^i\| \le \sum_{i=n+1}^{m} \|Q\|^i$$

Recall the geometric series sum $\displaystyle\sum_{i=n+1}^{m} a^i = \frac{a^{n+1}(1 - a^{m-n})}{1 - a}$. Then

$$\|T_m - T_n\| \le \frac{\|Q\|^{n+1}}{1 - \|Q\|}$$ which implies $\{T_0, T_1, \dots\}$ is Cauchy.

Now we show that $\displaystyle\sum_{i=0}^{\infty} Q^i$ is the right-inverse of $I - Q$.

$$\begin{aligned}(I - Q)\sum_{k=0}^{\infty} Q^k &= \sum_{k=0}^{\infty} Q^k - Q\sum_{k=0}^{\infty} Q^k \\ &= I + \sum_{k=1}^{\infty} Q^k - Q\sum_{k=0}^{\infty} Q^k = I\end{aligned}$$

Similarly, $\displaystyle\sum_{i=0}^{\infty} Q^i$ is the left-inverse of $I - Q$ also, and hence it is the inverse of $I - Q$.

The spectrum

Suppose $F \in \mathcal{L}(\mathcal{V})$. The *spectrum* of F is

$$\operatorname{spec}(F) = \bigl\{\lambda \in \mathbb{C} \;;\; (\lambda I - F) \text{ is not invertible}\bigr\}$$

The *spectral radius* of F is

$$\rho(F) = \max\bigl\{|\lambda| \;;\; \lambda \in \operatorname{spec}(F)\bigr\}.$$

We say λ is an *eigenvalue* of F if there exists $x \in \mathcal{V}$ such that

$$Fx = \lambda x$$

Clearly, if λ is an eigenvalue of F, then $\lambda \in \operatorname{spec}(F)$. But *the converse is not true in general*. In general

$$\bigl\{\lambda \in \mathbb{C} \;;\; \lambda \text{ is an eigenvalue of } F\bigr\} \subseteq \operatorname{spec}(F)$$

These sets are equal for finite-dimensional matrices.

The spectral radius and the norm

The spectral radius satisfies

$$\rho(F) \leq \|F\|$$

for all operators $F \in \mathcal{L}(\mathcal{V})$.

Proof

For matrices, one can see this by considering an eigenvector. But in general F may not have eigenvectors.

Suppose $|\lambda| > \|F\|$. Then, set $Q = \lambda^{-1}F$, and then $\|Q\| < 1$, which implies that $I - Q$ is invertible by the small-gain theorem.
Also, if $I - Q$ is invertible, so is $\lambda(I - Q)$, which is

$$\lambda(I - Q) = \lambda(I - \lambda^{-1}F) = \lambda I - F$$

Hence $\lambda \notin \operatorname{spec}(F)$.

The spectrum of a product

Consider operators $P \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $Q \in \mathcal{L}(\mathcal{V}, \mathcal{U})$. Then

$$(I - PQ) \text{ is invertible} \qquad \Longleftrightarrow \qquad (I - QP) \text{ is invertible}$$

Proof: If $I - PQ$ is invertible, we can construct the inverse of $I - QP$ according to

$$(I - QP)^{-1} = I + Q(I - PQ)^{-1}P$$

This is called the *Sherman-Morrison-Woodbury* formula, or the *Matrix-inversion lemma*. It can be shown directly by multiplying both sides by $I - QP$.

The spectrum of a product

An immediate consequence is that, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$,

$$\lambda \in \operatorname{spec}(PQ) \qquad \Longleftrightarrow \qquad \lambda \in \operatorname{spec}(QP)$$

Proof:

$$\begin{aligned} \lambda I - PQ \text{ is invertible} &\iff I - \lambda^{-1}PQ \text{ is invertible} \\ &\iff I - \lambda^{-1}QP \text{ is invertible} \\ &\iff \lambda I - QP \text{ is invertible} \end{aligned}$$

The spectrum of a product

Example

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad PQ = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

The adjoint operator

Suppose $\mathcal{V}$ and $\mathcal{Z}$ are Hilbert spaces, and $F \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$. The operator $F^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ is called the *adjoint* of F if

$$\langle z, Fv \rangle = \langle F^* z, v \rangle$$

for all $v \in \mathcal{V}$ and $z \in \mathcal{Z}$.

Properties

- $\|F^*\| = \|F\| = \|F^*F\|^{\frac{1}{2}}$
- $\|F\|^2 = \rho(F^*F)$.

Example

The adjoint of a matrix is the complex conjugate transpose.

Self-adjoint operators

The operator F is called *self-adjoint* or *hermitian* if $F = F^*$.

- If F is self-adjoint, then $\rho(F) = \|F\|$.
- The *quadratic form* $\langle Fv, v\rangle$ takes only real values.
- If $\lambda \in \operatorname{spec}(F)$, then $\lambda \in \mathbb{R}$.

Positive operators

A self-adjoint operator F is called *positive semidefinite*, written $F \geq 0$, if

$$\langle Fv, v\rangle \geq 0 \qquad \text{for all } v$$

A self-adjoint operator F is called *positive definite*, written $F > 0$, if

$$\text{there exists } \varepsilon > 0 \text{ such that} \qquad \langle Fv, v\rangle \geq \varepsilon \|v\|^2 \text{ for all } v$$

For matrices, this coincides with the usual definition of positive definiteness. If $F \in \mathbb{R}^{n\times n}$

$$F > 0 \qquad \Longrightarrow \qquad \langle Fv, v\rangle = v^* F v \geq \frac{\lambda_{\mathsf{min}}(F)}{2} v^* v$$

Isometric operators

The operator U is called *isometric* if $U^*U = I$.

Properties

- Angles are preserved: $\langle Uv_1, Uv_2\rangle = \langle U^*Uv_1, v_2\rangle = \langle v_1, v_2\rangle$
- Norms are preserved: $\|Uv\| = \|v\|$ for all v.
- Distances are preserved: $\|Uv_1 - Uv_2\| = \|v_1 - v_2\|$.

Unitary operators

The operator U is called *unitary* if

$$U^*U = I \qquad \text{and} \qquad UU^* = I$$

A unitary operator $U : \mathcal{U} \to \mathcal{V}$ is called an *isomorphism*.

Example

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{is isometric, but not unitary}$$

The L_2 spaces

The Hilbert space $L_2(-\infty, \infty)$ is the set of functions $u : \mathbb{R} \to \mathbb{C}^n$ with inner product

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(t)^* v(t)\, dt$$

The Hilbert space $\hat{L}_2(j\mathbb{R})$ is the set of functions $\hat{u} : j\mathbb{R} \to \mathbb{C}^n$ with inner product

$$\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{v}(j\omega)\, dt$$

The Fourier Transform

The Fourier transform is a map $\Phi : L_2(-\infty, \infty) \to \hat{L}_2(j\mathbb{R})$ defined by

$$\Phi : u \mapsto \hat{u} \qquad\qquad \hat{u}(j\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t}\, dt$$

- Φ is a bounded linear operator.
- Φ is invertible. The inverse is given by $u(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \hat{u}(j\omega) e^{j\omega t}\, d\omega$.
- Φ is unitary. It is an isomorphism between $L_2(-\infty, \infty)$ and $\hat{L}_2(j\mathbb{R})$.

Frequency domain spaces

The *open right-half plane* and *closed right half-plane* are

$$\mathbb{C}^+ = \Big\{ z \in \mathbb{C} \; ; \mathrm{Re}(z) > 0 \Big\} \qquad \text{and} \qquad \bar{\mathbb{C}}^+ = \Big\{ z \in \mathbb{C} \; ; \mathrm{Re}(z) \geq 0 \Big\}$$

The space H_2

The space H_2 is the set of functions $\hat{u} : \bar{\mathbb{C}}^+ \longrightarrow \mathbb{C}^n$ for which

- $\hat{u}$ is analytic in the open right-half plane $\mathbb{C}^+$.
- For almost every real number ω,

$$\hat{u}(j\omega) = \lim_{\sigma \to 0^+} \hat{u}(\sigma + j\omega)$$

- The maximum integral over a vertical line $\mathrm{Re}(z) = \sigma$ in $\bar{C}^+$

$$\sup_{\sigma \geq 0} \int_{-\infty}^{\infty} \|\hat{u}(\sigma + j\omega)\|_2^2 \, d\omega \qquad \text{is finite}$$

Rational functions

A rational function $\hat{u}$ is in H_2 if it is strictly proper and has no poles in the closed right-half plane.

The Laplace transform

The Laplace transform $\Lambda : u \mapsto \hat{u}$ is defined by

$$\hat{u}(s) = \int_0^\infty u(t)e^{-st}\,dt$$

Notes

- $\Lambda : L_2[0,\infty) \to H_2$
- Λ is a bounded linear operator.
- Λ is invertible. It is an isometric isomorphism between $L_2[0,\infty)$ and H_2.

The inner product in H_2

Given a function $\hat{u} \in H_2$, this defines a function on the imaginary axis which is an element of $\hat{L}_2(j\mathbb{R})$. We define the inner product of two functions in H_2 to be their inner product as elements of $\hat{L}_2(j\mathbb{R})$. That is

$$\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{u}^*(j\omega)\hat{v}(j\omega)\,d\omega$$

Note that H_2 is a subspace of $L_2(j\mathbb{R})$.

Summary of signal spaces

The signal spaces are

$$
\begin{array}{ccc}
L_2(-\infty,\infty) & \supset & L_2[0,\infty) \\
\text{Fourier } \Phi \Big\downarrow \Big\uparrow \Phi^{-1} & & \text{Laplace } \Lambda \Big\downarrow \Big\uparrow \Lambda^{-1} \\
\hat{L}_2(j\mathbb{R}) & \supset & H_2
\end{array}
$$

The space $\hat{L}_\infty(j\mathbb{R})$

Consider the set of matrix-valued functions

$$\hat{L}_\infty = \left\{ G : j\mathbb{R} \to \mathbb{C}^{p\times m} \; ; \; \|G\|_\infty = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}(\hat{G}(j\omega)) \text{ is finite} \right\}$$

We will use this as a space of *transfer functions*.

Multiplication operators

The function $G \in L_\infty(j\mathbb{R})$ defines a *multiplication operator* $M_{\hat{G}} : L_2(j\mathbb{R}) \to L_2(j\mathbb{R})$

$$\hat{y} = M_{\hat{G}}\hat{u} \qquad \Longleftrightarrow \qquad \hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$

Notes

- $\hat{G}$ is our usual notion of transfer function
- Using the Fourier transform, $\hat{G}$ defines a map $G : L_2(-\infty,\infty) \to L_2(-\infty,\infty)$ by

$$G = \Phi^{-1} M_{\hat{G}} \Phi$$

- If $\hat{G}$ is rational, then $\hat{G} \in \hat{L}_\infty(j\mathbb{R})$ if and only if it is proper and has no poles on the imaginary axis.

The shift operator

The shift operator $S_\tau : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ is defined by

$$y = S_\tau u \qquad \iff \qquad y(t) = u(t - \tau)$$

This is also called the τ-*delay*.

Time-invariance

An operator $G : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ is called *time-invariant* if

$$GS_\tau = S_\tau G \qquad \text{for all } \tau \geq 0$$

Theorem

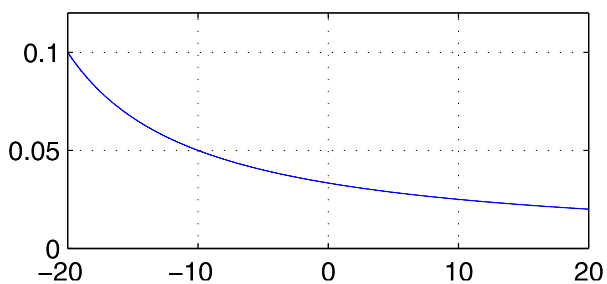
An operator $G : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ is time-invariant if and only there exists a function $\hat{G} \in \hat{L}_\infty(j\mathbb{R})$ such that the multiplication operator satisfies

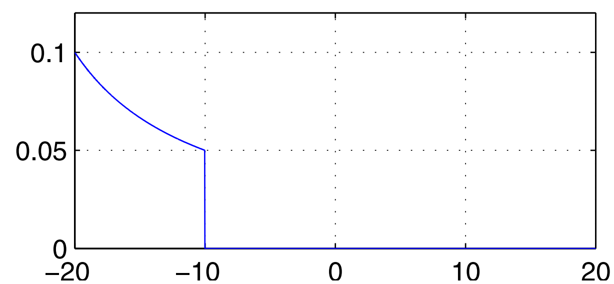
$$G = \Phi^{-1} M_{\hat{G}} \Phi$$

The truncation operator

The truncation operator $P_\tau : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ is defined by

$$y = P_\tau u \qquad \Longleftrightarrow \qquad y(t) = \begin{cases} u(t) & \text{for } t \le \tau \\ 0 & \text{otherwise} \end{cases}$$





Causality

The operator $G : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ is called *causal* if

$$P_\tau G P_\tau = P_\tau G \qquad \text{for all } \tau \in \mathbb{R}$$

Interpretation

$y_1 = P_\tau G u$ is the output signal on $(-\infty, \tau)$ corresponding to input u.
$y_2 = P_\tau G P_\tau u$ is the output signal on $(-\infty, \tau)$ corresponding to input $P_\tau u$.
If $y_1 = y_2$, then the output before time τ is unaffected by inputs after time τ.

Time-invariance and causality

If G is time-invariant, then it is causal if and only if

$$P_0 G P_0 = P_0 G$$

That is, we need only check the causality condition $P_\tau G P_\tau = P_\tau G$ at $\tau = 0$.

Corollary

A time-invariant operator G is causal if and only if

$$u \in L_2[0, \infty) \qquad \Longrightarrow \qquad Gu \in L_2[0, \infty)$$

Notes

- This follows from $P_0 G(I - P_0) = 0$.

The space H_∞

The set of matrix-valued functions $G : \bar{C}^+ \to \mathbb{C}^{p\times m}$ satisfying the following properties:

- $\hat{G}(s)$ is analytic in $\mathbb{C}^+$;
- For almost every real number ω

$$\lim_{\sigma\to 0^+} \hat{G}(\sigma + j\omega) = \hat{G}(j\omega)$$

- $\sup_{s\in\bar{\mathbb{C}}^+} \overline{\sigma}(\hat{G}(s))$ is finite.

Notes

- The norm on H_∞ is given by $\|G\|_\infty = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}(\hat{G}(j\omega))$
- If $\hat{G}$ is rational, then $\hat{G} \in H_\infty$ if and only if it is proper and has no poles in the closed right-half of the complex plane.

Theorem

- Every $\hat{G} \in H_\infty$ defines a causal, time-invariant operator $G : L_2[0, \infty) \to L_2[0, \infty)$.

  $z = Gu$ is defined by the multiplication operator

  $$\hat{z}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$

- If $G : L_2[0, \infty) \to L_2[0, \infty)$ is bounded, linear, and time-invariant, then there exists a function $\hat{G} \in H_\infty$ such that

  $$z = Gu \qquad \Longleftrightarrow \qquad \hat{z}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$$

Notes

- There is a one-to-one correspondence between functions in H_∞ and linear time-invariant (LTI) systems.

- We denote the subset of rational functions in H_∞ by RH_∞.

- Every function $\hat{G} \in RH_\infty$ can be expressed as

  $$\hat{G}(s) = C(sI - A)^{-1}B + D$$

  for some matrices A, B, C, D.