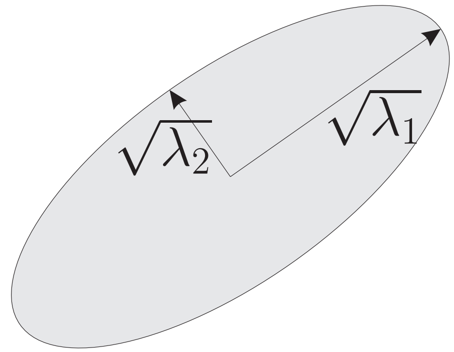
Engr210a Lecture 9: Controllability and Observability

- Ellipsoids
- The controllability gramian
- Lyapunov equations
- The observability gramian
- Controllability and observability ellipsoids
- Lyapunov stability

Ellipsoids



$$E = \left\{ x \in \mathbb{R}^n \; ; \; x^* Z^{-1} x \leq 1 \right\}$$

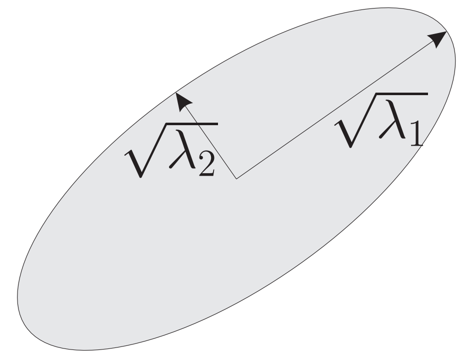
Notes

- $Z \in \mathbb{R}^{n \times n}$, $Z = Z^T$, $Z > 0$.
- semiaxis lengths: $\sqrt{\lambda_i}$, where λ_i are eigenvalues of Z
- semiaxis directions: eigenvectors of Z

Alternate representation of ellipsoids

Suppose $\mathcal{U}$ is a Hilbert space, $M : \mathcal{U} \to \mathbb{R}^n$, and $\mathrm{image}(M) = \mathbb{R}^n$. Define $Z = MM^*$. Then the following sets are the same ellipsoid.

- $E_1 = \{x \in \mathbb{R}^n \;;\; x^* Z^{-1} x \leq 1\}$
- $E_2 = \{Z^{\frac{1}{2}} y \;;\; y \in \mathbb{R}^n, \|y\|_2 \leq 1\}$
- $E_3 = \{Mu \;;\; u \in \mathcal{U}, \|u\|_2 \leq 1\}$



Proof

- Clearly $E_1 = E_2$; set $y = Z^{-\frac{1}{2}} x$.
- We show $E_3 \subset E_1$. Suppose $x \in E_3$, so $x = Mu$ for some u with $\|u\| \leq 1$. Then

$$x^* Z^{-1} x = \langle u, M^* Z^{-1} M u \rangle = \langle u, M^* (MM^*)^{-1} M u \rangle$$

Now notice that $P = M^*(MM^*)^{-1}M$ is a projection operator, that is $P^2 = P$ and $P^* = P$. Hence $\|P\| \leq 1$, and

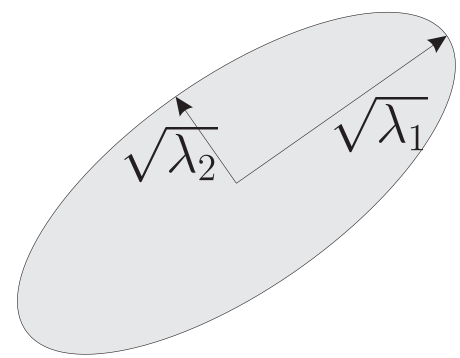
$$x^* Z^{-1} x = \langle u, Pu \rangle = \langle Pu, Pu \rangle = \|Pu\|^2 \leq \|u\|^2 \leq 1$$

so $x \in E_1$.

Alternate representation of ellipsoids, continued

Suppose $\mathcal{U}$ is a Hilbert space, $M : \mathcal{U} \to \mathbb{R}^n$, and $\mathrm{image}(M) = \mathbb{R}^n$. Define $Z = MM^*$. Then the following sets are the same ellipsoid.

- $E_1 = \left\{ x \in \mathbb{R}^n \ ; \ x^* Z^{-1} x \leq 1 \right\}$
- $E_2 = \left\{ Z^{\frac{1}{2}} y \ ; \ y \in \mathbb{R}^n, \|y\|_2 \leq 1 \right\}$
- $E_3 = \left\{ Mu \ ; \ u \in \mathcal{U}, \|u\|_2 \leq 1 \right\}$



Proof continued

- Conversely, we show $E_1 \subset E_3$. Suppose $x \in E_1$, so $x^* Z^{-1} x \leq 1$. Let

$$u = M^*(MM^*)^{-1} x$$

Then

$$Mu = MM^*(MM^*)^{-1} x = x$$

and

$$\|u\|^2 = \langle u, u \rangle == \langle x, (MM^*)^{-1} MM^* (MM^*)^{-1} x \rangle = x^* Z^{-1} x \leq 1$$

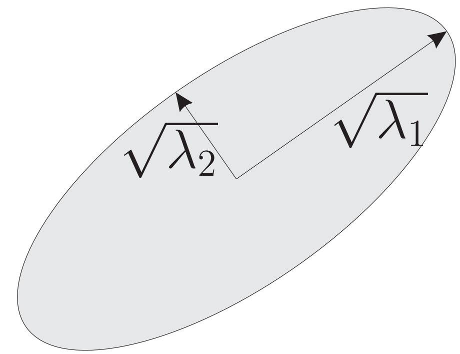
so $x \in E_3$.

- Aside: $\mathrm{image}(P) = (\ker(M))^\perp$ for the projection $P = M^*(MM^*)^{-1} M$.

Controllability and ellipsoids

The set of states reachable with an input $u \in L_2(-\infty, 0]$ with norm $\|u\| \leq 1$ is

$$\begin{aligned} E_c &= \left\{ \Psi_c u \ ; \ \|u\| \leq 1 \right\} \\ &= \left\{ \xi \in \mathbb{R}^n \ ; \ \xi^* X_c^{-1} \xi \leq 1 \right\} \end{aligned}$$



Notes

- The matrix $X_c = \Psi_c \Psi_c^*$ is called the *controllability gramian*.
- semiaxis lengths: $\sqrt{\lambda_i}$, where λ_i are eigenvalues of X_c
- semiaxis directions v_i are eigenvectors of X_c

Interpretation

- Directions v_i corresponding to large λ_i are *strongly controllable*.
- Directions v_i corresponding to small λ_i are *weakly controllable*.
- The energy required to drive the final state to $x \in \mathbb{R}^n$ is

$$\begin{aligned} \|u_{\text{opt}}\| &= \langle \Psi_c^* X_c^{-1} x, \Psi_c^* X_c^{-1} x \rangle \\ &= \langle X_c^{-1} x, x \rangle = x^* X_c^{-1} x \end{aligned}$$

Computing the adjoint operator

The controllability operator $\Psi_c : L_2(-\infty, 0] \to \mathbb{R}^n$ is defined by

$$\Psi_c u = \int_{-\infty}^{0} e^{-A\tau} B u(\tau)\, d\tau \qquad \text{for } u \in L_2(-\infty, 0]$$

The adjoint

By definition, $\Psi_c^* : \mathbb{R}^n \to L_2(-\infty, 0]$ is the unique operator that satisfies

$$\begin{aligned} \langle \Psi_c^* x, u \rangle &= \langle x, \Psi_c u \rangle \qquad \text{for all } u \in L_2(-\infty, 0] \text{ and } x \in \mathbb{R}^n \\ &= x^* \int_{-\infty}^{0} e^{-A\tau} B u(\tau)\, d\tau \\ &= \int_{-\infty}^{0} \left(B^* e^{-A^*\tau} x \right)^* u(\tau)\, d\tau \end{aligned}$$

Hence Ψ_c^* is defined by

$$(\Psi_c^* x)(t) = B^* e^{-A^* t} x$$

Computing the controllability gramian

Suppose A is Hurwitz. The controllability gramian $X_c \in \mathbb{R}^{n \times n}$ defined by $X_c = \Psi_c \Psi_c^*$ is given by

$$X_c = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau} \, d\tau$$

Proof

We have

$$\Psi_c u = \int_{-\infty}^0 e^{-A\tau} B u(\tau) \, d\tau \qquad\qquad (\Psi_c^* x)(t) = B^* e^{-A^* t} x$$

for all $u \in L_2(-\infty, 0]$ and $x \in \mathbb{R}^n$. Hence

$$X_c x = \Psi_c \Psi_c^* x = \int_{-\infty}^0 e^{-A\tau} B B^* e^{-A^*\tau} x \, d\tau \qquad \text{for all } x \in \mathbb{R}^n$$

which implies

$$X_c = \int_{-\infty}^0 e^{-A\tau} B B^* e^{-A^*\tau} \, d\tau$$

Computing the controllability gramian

Suppose A is Hurwitz. The controllability gramian $X_c \in \mathbb{R}^{n\times n}$ is the unique solution to the linear equation

$$AX_c + X_c A^* + BB^* = 0$$

This equation is called a *Lyapunov* equation.

Notes

- This is a linear equation, hence it is easily solvable. We can just rewrite it in the form

$$Px = q$$

  where $x \in \mathbb{R}^{\frac{n(n+1)}{2}}$ is a vector whose components are the $n(n+1)/2$ distinct entries of X.

- The matrix $X_c \geq 0$, since it is given by $X_c = \Psi_c \Psi_c^*$.

- If (A, B) is controllable, then $X_c > 0$, from our previous result that

$$\ker(\Psi_c^*) = \ker(\Psi_c \Psi_c^*)$$

Lyapunov equations

Suppose A and Q are square matrices, and A is Hurwitz. Then

$$X = \int_0^\infty e^{At} Q e^{A^*t}\, dt$$

is the unique solution to the Lyapunov equation

$$AX + XA^* + Q = 0$$

Proof

- Note that the integral converges, since A is Hurwitz implies e^{At} decays exponentially.

$$\frac{d}{dt}\Big(e^{At} Q e^{A^*t}\Big) = A\Big(e^{At} Q e^{A^*t}\Big) + \Big(e^{At} Q e^{A^*t}\Big)A^*$$

$$\Longrightarrow \qquad \int_0^\infty \frac{d}{dt}\Big(e^{At} Q e^{A^*t}\Big)\, dt = A\int_0^\infty e^{At} Q e^{A^*t}\, dt + \int_0^\infty e^{At} Q e^{A^*t}\, dt A^*$$

$$\Longrightarrow \qquad -Q = AX + XA^*$$

- *Uniqueness:* This equation defines a linear map $\Pi : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, where $\Pi(X) = -Q$. Then $\mathrm{image}(\Pi) = \mathbb{R}^{n^2}$ implies $\ker(\Pi) = \{0\}$.

Summary of controllability

- If A is Hurwitz, the controllability gramian

$$X_c = \int_0^\infty e^{A\tau} B B^* e^{A^*\tau}\, d\tau$$

  is a real, symmetric matrix, and $X_c \geq 0$

- $X_c = \Psi_c \Psi_c^*$.

- We can compute X_c; it is the unique solution to $AX_c + X_c A^* + BB^* = 0$.

- The eigenvalues of X_c provide information on how controllable the system is. If any $\lambda_i = 0$, the system is not controllable.

Singular value interpretation

- The eigenvalues of $\Psi_c \Psi_c^*$ are the squares of the singular values of Ψ_c.

- This fits with our standard notion of rank; instead of looking at $\mathrm{rank}(C_{AB})$ to determine $\mathrm{image}(\Psi_c)$, look at the singular values of Ψ_c.

Finite-time controllability

- Consider the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = 0$$

- The same approach works on the finite-time interval, with

$$x(t) = \Upsilon_t u = \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

- The finite-time controllability gramian is

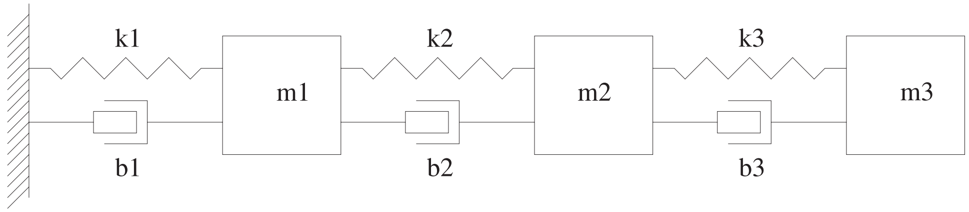
$$X_t = \int_0^t e^{A\tau} BB^* e^{A^*\tau}\, d\tau$$

- $X_t \geq X_s$ if $t \geq s$. Hence

$$x^* X_t^{-1} x \leq x^* X_s^{-1} x \qquad \text{if } t \geq s$$

That is, it takes less energy to reach a state x over a longer time.

Example

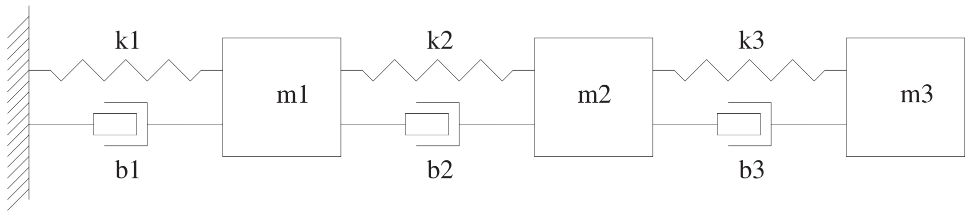


- Masses $m_i = 1$, spring constants $k = 1$, damping constants $b = 0.8$.
- Equations of motion $\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$

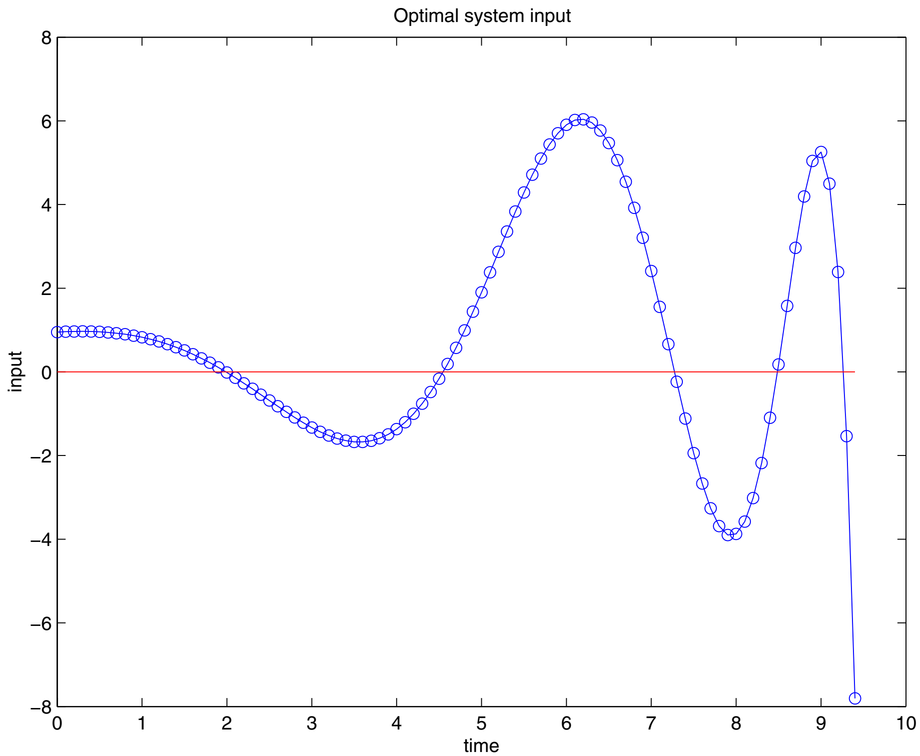
$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & 0.8 & 0 \\ 1 & -2 & 1 & 0.8 & -1.6 & 0.8 \\ 0 & 1 & -1 & 0 & 0.8 & -0.8 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

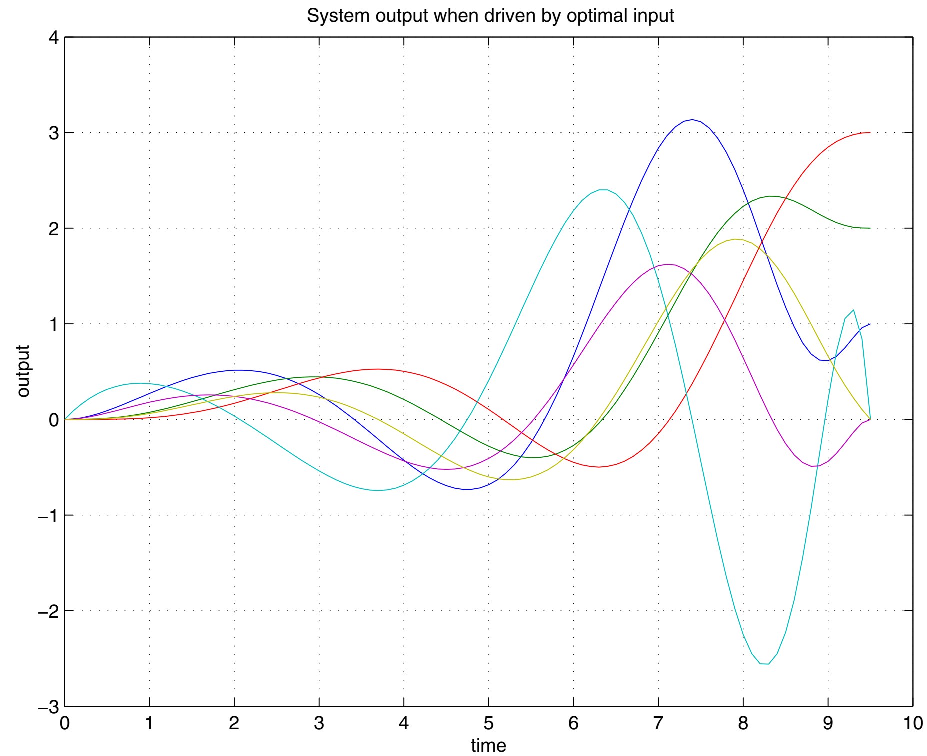
- Input u is a force applied to the first mass. States x_1, x_2, x_3 are displacements, states x_4, x_5, x_6 are velocities.
- Desired state is $\xi = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$, in 9.5 seconds.

Example



Optimal input: $u_{\text{opt}} = \Upsilon_t^* X_t^{-1} \xi$.





Observability

- Suppose we have a stable state-space system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{with initial condition } x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

- The solution is $y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$

- This defines a map $\Psi_o : \mathbb{R}^n \to L_2[0,\infty)$ by

$$y = \Psi_o x_0 + \Lambda_o u$$

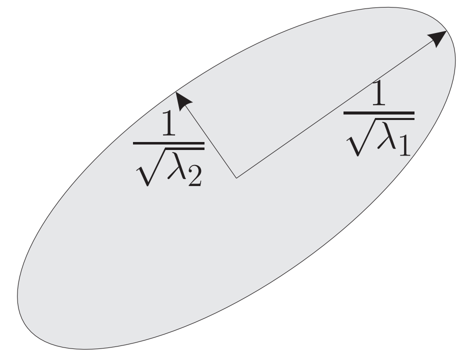
- We know which states are unobservable:

$$\ker(\Psi_o) = \ker \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- How *observable* is a particular state? Given $x \in \mathbb{R}^n$, we will compute $\|\Psi_o x\|$.

More ellipsoids

Suppose $\mathcal{U}$ is a Hilbert space, $M : \mathbb{R}^n \to \mathcal{U}$, and $\ker(M) = \{0\}$. Define $Y = M^*M$. Then the following sets are the same ellipsoid.



- $O_1 = \left\{ x \in \mathbb{R}^n \; ; \; x^*Yx \leq 1 \right\}$
- $O_2 = \left\{ x \in \mathbb{R}^n \; ; \; \|Mx\| \leq 1 \right\}$

Notes

- semiaxis lengths: $\frac{1}{\sqrt{\lambda_i}}$, where λ_i are eigenvalues of Y
- semiaxis directions: eigenvectors of Y
- The directions of the axes of this ellipsoid are the same as those of

$$\left\{ x \in \mathbb{R}^n \; ; \; x^*Y^{-1}x \leq 1 \right\}$$

  but the magnitudes are inverted.

Observability

- Given $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|\Psi_o x\| &= \langle \Psi_o x, \Psi_o x \rangle \\ &= \langle x, \Psi_o^* \Psi_o x \rangle \\ &= x^* Y_o x \end{aligned}$$

  where $Y_o = \Psi_o^* \Psi_o$. The matrix Y_o is called the *observability gramian*.

- The set of initial states which result in an output y with norm $\|y\| \leq 1$ is given by the ellipsoid

$$\begin{aligned} E_o &= \left\{ x \in \mathbb{R}^n \ ; \ \|\Psi_o x\| \leq 1 \right\} \\ &= \left\{ x \in \mathbb{R}^n \ ; \ x^* Y_o x \leq 1 \right\} \end{aligned}$$

  Note that the major axis corresponds to *weakly observable* states.

Caveat

- Some authors plot the ellipsoid

$$\left\{ x \in \mathbb{R}^n \ ; \ x^* Y^{-1} x \leq 1 \right\}$$

  so that the major axes correspond to *strongly observable* states.

Summary of observability

- Results parallel those of controllability.
- $Y_o = \Psi_o^* \Psi_o$ is the *observability gramian*.
- If A is Hurwitz, computation of the adjoint gives

$$Y_o = \int_0^\infty e^{A^* \tau} C^* C e^{A\tau} \, d\tau$$

  which is real, symmetric, and $Y_o \geq 0$.
- We can compute Y_o; it is the unique solution to

$$A^* Y_o + Y_o A + C^* C = 0$$

- Compare this with the Lyapunov equation for the controllability gramian

$$A X_c + X_c A^* + B B^* = 0$$

- The eigenvalues of Y_o provide information on how observable the system is. If any $\lambda_i = 0$, the system is not observable.
- If (C, A) is observable then $Y_o > 0$.

Lyapunov theory

Suppose $Q > 0$. Then A is Hurwitz if and only if there exists a positive definite solution $X > 0$ to the Lyapunov equation

$$A^*X + XA + Q = 0$$

Notes

- This provides the converse to our earlier results.

Proof

only if: Since A is Hurwitz, we know the unique solution is given by

$$X = \int_0^\infty e^{A^*\tau} Q e^{A\tau}\, d\tau$$

This is positive, since e^{At} is invertible for all t.

if: Suppose $X > 0$ satisfies the Lyapunov equation. Then

$$0 = v^*(A^*X + XA + Q)v = \lambda^* v^* X v + \lambda v^* X v + v^* Q v$$

Since $v^* X v > 0$ we have

$$2\,\mathrm{Re}(\lambda) = -\frac{v^*Qv}{v^*Xv} < 0$$

Lyapunov theory

Suppose we have the system of ordinary differential equations

$$\dot{x}(t) = f(x)$$

where $x(t) \in \mathbb{R}^n$ and $f(0) = 0$. Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(x) > 0$ for $x \neq 0$

(iii) $\displaystyle \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) < 0$ for $x \neq 0$.

(iv) If $\{x_0, x_1, \dots\}$ is a sequence such that $\|x_k\| \to \infty$, then $V(x_k) \to \infty$.

Then the origin $x = 0$ is globally asymptotically stable. That is, for any initial condition

$$\lim_{t \to \infty} x(t) = 0$$

Lyapunov stability of linear systems

The function $V(x) = x^*Xx$ is a Lyapunov function for the linear system $\dot{x}(t) = Ax(t)$, since

$$\begin{aligned}\frac{d}{dt}V(x) &= \dot{x}^*(t)Yx(t) + x^*(t)Y\dot{x}(t)\\ &= x^*(t)(A^*X + XA)x(t)\\ &= -x^*(t)Qx(t) < 0\end{aligned}$$

Notes

- Hence, if X_c is the controllability gramian, the function $V(x) = x^*X_cx$ is a Lyapunov function for $\dot{x}(t) = Ax(t)$.
- Similarly, if Y_o is the controllability gramian, the function $V(x) = x^*Y_ox$ is a Lyapunov function for $\dot{x}(t) = A^*x(t)$.
- Corollary: The LMI condition

  $$A \text{ is Hurwitz} \quad \Longleftrightarrow \quad \text{there exists } X > 0 \text{ such that } A^*X + XA < 0$$

- There are many other interpretations for the gramians; e.g. as Lagrange multipliers, separating hyperplanes, storage functions, solutions to H-J equations, state covariance for systems driven by white noise, . . .